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A NOTE ON WEAK LIFTING MODULES

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Abstract

It is proved that the notions weak lifting and weak $\oplus$-supplemented are same notions.

1 Introduction

Throughout the paper all rings have identity and all modules are unitary. In [3] and [4], authors defined weak lifting modules: A module M is said to be a *weak lifting*, if for each semisimple submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $M_2 \cap N \ll M_2$. In both papers, they have posed many results but some of these results are actually redundant.

In [3], it is defined that a module M is called *weak $\oplus$-supplemented*, if for each semisimple submodule N of M, there exists a direct summand K of M such that $M = N + K$ and $N \cap K \ll K$.

Authors considered these two definitions in different manners, however these notions are the same notions.

Lemma 1.1. *Let N and L be two submodules of M so that $M = N + L$, also let N be semisimple, then $M = X \oplus L$ for some semisimple submodule X of N.*

Proof Obviously, $N \cap L$ is a direct summand of N, because N is semisimple. So, $N = (N \cap L) \oplus A$ for some submodule A of N. Then $M = [(N \cap L) \oplus A] + L = A + L$ but $A \cap L = (N \cap L) \cap A = 0$. So, $M = A \oplus L$. □

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Theorem 1.2. *For an R-module M, the following statements are equivalent:*

1. *Every semisimple submodule of M has a supplement.*
2. *M is weak $\oplus$-supplemented.*
3. *M is weak lifting.*

Proof (1 $\Rightarrow$ 2) Let N be a semisimple submodule of M. By assumption, N has a supplement L in M such that $M = N + L$ and $N \cap L \ll L$. By Lemma 1.1, L is a direct summand of M.

(2 $\Rightarrow$ 3) Let M be weak-$\oplus$-supplemented module and N be a semisimple submodule of M. Then N has a direct summand supplement L in M, i.e., $M = N + L = L \oplus L'$ and $N \cap L \ll L$ for some submodule L' of M. By Lemma 1.1, $M = X \oplus L$ where $X \leq N$ and we already have $L \cap N << L$. Hence M is weak lifting.

(3 $\Rightarrow$ 1) Let N be a semisimple submodule of M. By assumption, there exists a decomposition K of M such that $M = K \oplus K'$, $K \leq N$ and $N \cap K' \ll K'$ for some submodule K' of M. Then $M = N + K'$. □

The following result is proved in ([3], Lemma 2.4) by help of an extra property, namely (D_3) property, however (D_3) property is not necessary. We shall give the definition of (D_3) property for completeness:

A module M is said to have (D_3) property, if whenever M_1 and M_2 are direct summands in M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

Proposition 1.3. *Any direct summand of a weak lifting module is weak lifting.*

Proof Let M be a weak lifting module and N be a direct summand of M so that $M = N \oplus K$ for some submodule K of M. Let S be a semisimple submodule of N. Since $S \subseteq M$, then $M = S + T$ and $S \cap T \ll T$ for some submodule T of M. By the modular law, $N = S + (N \cap T)$. Then by Lemma 1.1, $M = S' \oplus T$ for some $S' \subseteq S$. By the modular law again, we get $N = S' \oplus (N \cap T)$, that is, $N \cap T$ is a direct summand of N. We claim that $T = (N \cap T) \oplus K$: Since $S' \subseteq N$ and $M = N \oplus K = S' \oplus T$, then K can be considered as a submodule of T, then $(N \cap T) \oplus K \subseteq T$. Conversely, if $t \in T \subseteq M = N \oplus K$, then $t = n + k$ for some $n \in N$ and $k \in K$, since $n = t - k \in T$, then $n \in N \cap T$. Hence $N \cap (S \cap T) \subseteq S \cap T \ll T = (N \cap T) \oplus K$. Since $S \cap T \subseteq N \cap T$, then by ([5], 19.3.(5)), $S \cap (N \cap T) \ll N \cap T$. □

In ([3], Theorem 2.7), authors prove that finite direct sum of w-$\oplus$-supplemented modules is w-$\oplus$ - supplemented. And by Theorem 1.2, we can say that finite direct sum of weak lifting modules are weak lifting. By considering with

Proposition 1.3, in [4], Proposition 2.1, Proposition 2.4 and Theorem 2.2 are unnecessary.

In [3], Proposition 2.3, Lemma 2.4, Theorem 2.8 are redundant and (D_3) property is not necessary in Proposition 2.6.

The following Lemma shows that, we should be careful about speaking direct summandness of a semisimple submodule.

Lemma 1.4. *Let M be a module with $RadM \trianglelefteq M$. Then there is no nonzero semisimple direct summand of M.*

Proof Let N be a semisimple direct summand of M, then $M = N \oplus K$ for some submodule K of M. Then by ([5], 21.6) $RadM = RadK$ and since $RadM \trianglelefteq M$, then $K \trianglelefteq M$ by ([1], Proposition 5.16). If $N \neq 0$, $K = M$, impossible. Therefore $N = 0$. □

As a result of above Lemma, we may say, for instance, over a domain R, divisible R-modules have no nonzero semisimple direct summands.

Therefore, in [4], Proposition 1.7, Theorem 1.8, should have been written more carefully.

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