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THE NEGATIVE INNER PRODUCT SETS

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Abstract

In this article we will define the *negative inner product sets* and characterize their properties. One property concerning the negative linear combination leads to the existence of the non-negative and positive solutions of some classes of systems of linear equations. Applications on obtuse cones and monotone matrices are also discussed.

1. Introduction

Let V be a vector space over the field of real numbers. An inner product $\langle \cdot, \cdot \rangle$ is a real number that satisfies the properties:

1. $\langle \alpha, \beta \rangle + \langle \gamma, \beta \rangle = \langle \alpha + \gamma, \beta \rangle$
2. $\langle k\alpha, \beta \rangle = k \langle \alpha, \beta \rangle$
3. $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$
4. $\langle \alpha, \alpha \rangle > 0,$ for $\alpha \neq 0.$

Two vectors α and β are said to be orthogonal if $\langle \alpha, \beta \rangle = 0$. A subset of a vector space is called an orthogonal set if $\langle \alpha, \beta \rangle = 0$ for all α and β in the set for which $\alpha \neq \beta$. An orthogonal set is independent. In an n-dimensional vector space, an independent (orthogonal) subset can have at most n vectors.

In this research, we are interested in a set which we called a negative inner

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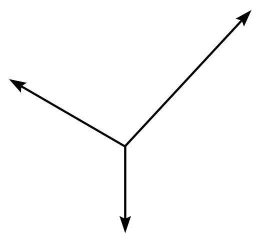
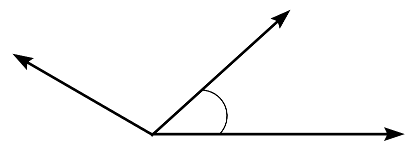
product set; that is, the set two different elements of which the inner product is negative. The negative inner product sets have similar properties to that of the orthogonal sets.



A negative inner product set



Not a negative inner product set

Definition 1.1 *Let W be a non-empty subset of a vector space over a field of real number. W is called a negative inner product (NIP) set if $\langle \alpha, \beta \rangle < 0$ for all α and β in W, with $\alpha \neq \beta$.*
Examples of NIP sets are as follows:

1. The singleton set with a non-zero vector.
2. $\{\alpha, -\alpha\}$ where α is a non-zero vector,
3. $\{\alpha, \beta, \gamma\}$, where $\alpha = \rho$, $\beta = -\rho - \dfrac{\langle \rho, \theta \rangle}{|\langle \rho, \theta \rangle|}\theta$ and $\gamma = -\dfrac{\rho}{|\rho|} - \dfrac{\theta}{|\theta|}$, where ρ and θ are independent vectors.

The properties of an NIP set are different but similar to that of an orthogonal set:

1. In an n-dimensional vector space, there are at most $n + 1$ vectors in the NIP set.
2. A proper subset of an NIP set is independent.

Note :

1. The independent set may not be an NIP set.
   The NIP set may not be independent.
   There exists an NIP set that has n+1 elements.

2. Since the NIP set can be defined only in real vector spaces, so, throughout this article the vector spaces (finite or infinite dimensional) means the vector spaces over the field of real numbers.

2. Main Results

Lemma 2.1 *Let m be a positive integer, and $\alpha_1, \alpha_2, \ldots, \alpha_m$ be non-zero vectors in a vector space. Also, let*

$$\beta = \sum_{i=1}^{m} k_i \alpha_i \neq 0$$

where $k_i < 0$ for all $i \in I = \{1, 2, \ldots, m\}$. Then there is α_s for some $s \in I$ such that $\langle \alpha_s, \beta \rangle < 0$

Proof. Suppose that $\langle \alpha_i, \beta \rangle \geq 0$ for all $i \in I$. Then

$$\langle \beta, \beta \rangle = \left\langle \sum_{i=1}^{m} k_i \alpha_i, \beta \right\rangle = \sum_{i=1}^{m} k_i \langle \alpha_i, \beta \rangle \leq 0,$$

since $k_i < 0$ for all $i \in I$. This is a contradiction since $\langle \beta, \beta \rangle > 0$. Thus, there must be α_s for some $s \in I$ such that $\langle \alpha_s, \beta \rangle < 0$. □

Lemma 2.2 *Let m be a positive integer, and $\alpha_1, \alpha_2, \ldots, \alpha_m$ be non-zero vectors in a vector space. If $k_i, i \in I = \{1, 2, \ldots, m\}$ are real numbers such that $\sum\limits_{i=1}^{m} k_i \alpha_i \neq 0$. If $\left\langle \alpha_j, \sum\limits_{i=1}^{m} k_i \alpha_i \right\rangle \leq 0$ for all $j \in I$, then there is $s \in I$ such that $k_s < 0$.*

Proof. Let $\beta = \sum\limits_{i=1}^{m} k_i \alpha_i \neq 0$. Suppose that $k_j \geq 0$ for all $j \in I$. Since $\langle \alpha_j, \beta \rangle \leq 0$ for all j, then $\langle \beta, \beta \rangle = \left\langle \sum\limits_{i=1}^{m} k_i \alpha_i, \beta \right\rangle = \sum\limits_{i=1}^{m} k_i \langle \alpha_i, \beta \rangle \leq 0$. This is a contradiction since $\langle \beta, \beta \rangle > 0$. Thus, there is $s \in I$ such thak $k_s < 0$. □

Lemma 2.3 (Gordan's Theorem)
Let A be an $m \times n$ matrix whose elements are real numbers. One and only one of the following two systems has solutions: (i) $A\boldsymbol{x} < 0$ for $\boldsymbol{x} \in R^{n \times 1}$; (ii)

$A^T\boldsymbol{y} = 0, \boldsymbol{y} \geq \mathbf{0}$ for some non-zero $\boldsymbol{y} \in R^{m\times 1}$.
(For the proof see [2])

Lemma 2.4 *Let m be a positive integer, and $W = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a subset of a vector space. If W is independent, then there are negative real numbers k_i for $i \in I = \{1, 2, \ldots, m\}$ such that $\left\langle \alpha_j, \sum_{i=1}^{m} k_i\alpha_i \right\rangle < 0$ for all $j \in I$.*

Proof. Consider the system of linear inequalities:

$$\begin{array}{ccccccccc}
\langle\alpha_1,\alpha_1\rangle k_1 & + & \langle\alpha_1,\alpha_2\rangle k_2 & + & \cdots & + & \langle\alpha_1,\alpha_m\rangle k_m & < & 0 \\
\langle\alpha_2,\alpha_1\rangle k_1 & + & \langle\alpha_2,\alpha_2\rangle k_2 & + & \cdots & + & \langle\alpha_2,\alpha_m\rangle k_m & < & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
\langle\alpha_m,\alpha_1\rangle k_1 & + & \langle\alpha_m,\alpha_2\rangle k_2 & + & \cdots & + & \langle\alpha_m,\alpha_m\rangle k_m & < & 0
\end{array}$$

We are going to show that this system of inequalities has a negative solution. Let $B = [\langle\alpha_i, \alpha_j\rangle]_{i,j=1}^{m}$ be the coefficient matrix of the system and $\boldsymbol{k} = [k_i]_{i=1}^{m}$. Note that $B^T = B$. It can be shown that $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is independent implies B is invertible. Hence the system $B\boldsymbol{k} < 0$ has a solution $\boldsymbol{k}$. Also $\boldsymbol{u}^T B\boldsymbol{u} > 0$ for any non-zero vector $\boldsymbol{u} \in R^{m\times 1}$. Now, let $A^T = [B \,|\, I_m]$ where I_m is the identity matrix of order m, and $\boldsymbol{y}^T = \left[\boldsymbol{u}^T|\boldsymbol{v}^T\right]$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are m (column) vectors. So, $A^T\boldsymbol{y} = B\boldsymbol{u} + \boldsymbol{v}$. Consider the system (ii) $B\boldsymbol{u} + \boldsymbol{v} = \mathbf{0}, \boldsymbol{u} \geq \mathbf{0}$ and $\boldsymbol{v} \geq \mathbf{0}$. Suppose that this system has a solution $\boldsymbol{u} \geq \mathbf{0}$ and $\boldsymbol{v} \geq \mathbf{0}$, then $\boldsymbol{u}^T B\boldsymbol{u} + \boldsymbol{u}^T\boldsymbol{v} = \mathbf{0}$. This is impossible since $\boldsymbol{u}^T B\boldsymbol{u} > \mathbf{0}$ and $\boldsymbol{u}^T\boldsymbol{v} \geq \mathbf{0}$, and so the system (ii) has no solution. By Lemma 2.3 the system $\begin{bmatrix} B \\ I_m \end{bmatrix} \boldsymbol{x} < 0$ has a solution. Therefore there exists $\boldsymbol{k} < 0$ such that $B\boldsymbol{k} < 0$ □

Note : The vector $\boldsymbol{k}$ in Lemma 2.4 may not be negative. But if W is an NIP set, then $\boldsymbol{k}$ has to be negative.

Theorem 2.5 *Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be an NIP subset of a vector space. Let $k_i, i = 1, 2, \ldots, m$, be real numbers such that $\beta = \sum_{i=1}^{m} k_i\alpha_i \neq 0$. If $\langle\alpha_i, \beta\rangle \leq 0$ for all $i = 1, 2, \ldots, m$, then $k_i < 0$ for all $i = 1, 2, \ldots, m$.*

Proof. WLOG, suppose that $k_i \geq 0$ for all $i \in I = \{1, 2, ..., p\}$ and $k_i < 0$ for all $i \in J = \{p + 1, p + 2, ..., m\}$. (By Lemma 2.2 there is $k_i < 0$, for some i.) Consider $\langle\beta, \alpha_j\rangle = \sum_{i=1}^{p} k_i \langle\alpha_i, \alpha_j\rangle + \sum_{i=p+1}^{m} k_i \langle\alpha_i, \alpha_j\rangle \leq 0$.

Since $\langle\alpha_i, \alpha_j\rangle < 0$ for all $i \neq j$, then for $j \leq p$, we have $\sum_{i=p+1}^{m} k_i \langle\alpha_i, \alpha_j\rangle > 0$.

Then $\sum_{i=1}^{p} k_i \langle\alpha_i, \alpha_j\rangle < 0$ for $j \in I$.

Since $\sum_{i=1}^{p} k_i \langle \alpha_i, \alpha_j \rangle = \left\langle \sum_{i=1}^{p} k_i \alpha_i, \alpha_j \right\rangle < 0$, for all $j \in I$, then, by Lemma 2.2, there is $j \in I$ such that $k_j < 0$. This contradicts the assumption that $k_i \geq 0$ for all $i \in I$.
Therefore $k_i < 0$ for all $i = 1, 2, \ldots, m$. $\square$

Corollary *Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be an NIP subset of a vector space. Let $k_i, i = 1, 2, \ldots, m$, be real numbers, and $\beta = \sum_{i=1}^{m} k_i \alpha_i$. If $\langle \alpha_i, \beta \rangle > 0$ for all $i = 1, 2, \ldots, m$, then $k_i > 0$ for all $i = 1, 2, \ldots, m$.*

Theorem 2.6 *Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be an NIP subset of a vector space. If S is linearly dependent, then there is no vector β such that $\langle \alpha_i, \beta \rangle < 0$ for all $i = 1, 2, \ldots, m$.*

Proof. Since S is linearly dependent, then there are real numbers $k_i, i = 1, 2, \ldots, m-1$, not all zeros, such that $\alpha_m = \sum_{i=1}^{m-1} k_i \alpha_i$.
Since $\langle \alpha_j, \alpha_m \rangle = \left\langle \alpha_j, \sum_{i=1}^{m-1} k_i \alpha_i \right\rangle < 0$ for all $j = 1, 2, \ldots, m-1$, then $k_i < 0$ for all $i = 1, 2, \ldots, m-1$.
Suppose that $\langle \alpha_i, \beta \rangle < 0$ for $i = 1, 2, \ldots, m-1$.
Consider

$$\begin{aligned} \langle \alpha_m, \beta \rangle &= \left\langle \sum_{i=1}^{m-1} k_i \alpha_i, \beta \right\rangle = \sum_{i=1}^{m-1} k_i \langle \alpha_i, \beta \rangle \\ &> 0 \quad \text{(Since } k_i < 0 \text{ for all } i = 1, 2, \ldots, m-1) \end{aligned}$$

Therefore, there is no vector β such that $\langle \alpha_i, \beta \rangle < 0$ for all $i = 1, 2, \ldots, m$. $\square$

It follows corollaries:

Corollary 1 *If S is a set that has a dependent proper subset, then S is not an NIP set.*

Corollary 2 *A proper subset of an NIP set is independent.*

Corollary 3 *In a n-dimensional vector space, there are at most $n + 1$ vectors in an NIP set.*

Corollary 4 *Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be an NIP subset of a vector space. Then S is linearly independent if and only if there are negative numbers $k_i, i = 1, 2, \ldots, m$, such that $\left\langle \alpha_j, \sum_{i=1}^{m} k_i \alpha_i \right\rangle < 0$ for all $j = 1, 2, \ldots, m$.*

Proof: Follows from Lemma 2.4 and Corollary 2. $\square$

3. Applications

3.1 Obtuse Cone

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a subset of a vector space, we define the following notations:

Lin $S = \{\beta \mid \beta = \sum_{i=1}^{m} k_i\alpha_i,\ k_i \in R\}$, the linear subspace generated by S

cone $S = \{\beta \mid \beta = \sum_{i=1}^{m} k_i\alpha_i,\ k_i \geq 0\}$, the cone generated by S.

If $C \subseteq R^n$, we define the dual of C:

$$C^* = \{\boldsymbol{z} \in R^n \mid \langle \boldsymbol{x}, \boldsymbol{z} \rangle \leq 0, \forall \boldsymbol{x} \in C\}.$$

A non-empty subset C of R^n is called a cone if $x \in C$ implies $\lambda x \in C$ for all $\lambda \geq 0$. Thus, If $S \subseteq R^n$ then cone S defined above is a cone. A cone C is said to be acute if $\langle x, y \rangle \geq 0$ for all $x, y \in C$. A cone C is said to be obtuse if $C^* \cap$ Lin C is an acute cone.

If A is a $m \times n$ real matrix, then Lin A shall mean the linear subspace generated by columns of A, and cone A means the cone generated by columns of A.

The matrix A^TA is called the Gram matrix of A.

If A is a matrix with full column rank, then the matrix

$$A^+ = (A^TA)^{-1}A^T$$

is called the (Moore-Penrose) pseudoinverse of A.

A matrix G is said to be monotone if $G\boldsymbol{x} \geq 0$ implies $\boldsymbol{x} \geq 0$.

The following property concerning the introduced objects is well known (see [1], [4]).

Lemma 3.1 *Let A have full column rank. The following conditions are equivalent:*

1. cone A is obtuse,
2. cone A^{+T} is acute,
3. (cone A)$^* \cap$ Lin $A \subseteq -$cone A,
4. $(A^TA)^{-1} \geq 0$
5. A^TA is monotone.

To check that a matrix has one of these properties may be difficult. We have sufficient conditions to have these properties.

Theorem 3.2 *Let A be a matrix with full column rank. If the set of columns of A is an NIP set, then cone A is obtuse.*

Proof: We shall prove that the conditions in the theorem imply (iii) in Lemma 3.1.
Let $\boldsymbol{x} \in (\text{cone}\, A)^* \cap \text{Lin}\ A$. Then $\boldsymbol{x} = \sum_{i=1}^{m} k_i \boldsymbol{x}_i$, where k_i are real number, and $\boldsymbol{x}_i$, $i \in I = \{1, 2, \ldots, m\}$, are columns of A. Also, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 0$ for all $\boldsymbol{y} \in$ cone A. Thus, since $x_i \in$ cone A, for all $i \in I$, we have

$$\left\langle \sum_{i=1}^{m} k_i \boldsymbol{x}_i, \boldsymbol{x}_j \right\rangle \leq 0, \quad \text{for all } j \in I$$

Since $\{\boldsymbol{x}_j \mid j \in I\}$ is an NIP set, then, by Theorem 2.5, $k_i < 0$ for all $i \in I$.
Then, $\quad \boldsymbol{x} = \sum_{i=1}^{m} k_i \boldsymbol{x}_i = -\sum_{i=1}^{m} (-k_i)\boldsymbol{x}_i \in -\text{cone}\ A$, since $-k_i > 0$.
Therefore $\quad (\text{cone}A)^* \cap$ Lin $A \subseteq -$cone A
So, cone A is obtuse. □

3.2 Non-negative and positive solutions of system of equations.
Let A be a $m \times n$ matrix and $\boldsymbol{b}$ is an n-vector. The system of linear equations

$$A\boldsymbol{x} = \boldsymbol{b}$$

has a solution if rank $[A \mid \boldsymbol{b}] =$ rank A. If A is square, then the system has a unique solution if and only if A is non-singular (A is invertible). Here we are interested in the system in which the solution is non-negative. One obvious result is from the monotone property of the coefficient matrix A.

Lemma 3.3 *Let A be a square matrix. Then A is monotone if and only if A is invertible and $A^{-1} \geq 0$.*

Proof (see [4])

Lemma 3.4 *If A is invertible and $A^{-1} \geq 0$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has a non-negative solution if $\boldsymbol{b} \geq \boldsymbol{0}$.*

Proof: Since $A^{-1} \geq 0$ and $\boldsymbol{b} \geq \boldsymbol{0}$, then the solution $\boldsymbol{z} = A^{-1}\boldsymbol{b} \geq \boldsymbol{0}$. □

Corollary *If a matrix A has full column rank and the columns of A form an NIP set, then the system $A^T A\boldsymbol{x} = \boldsymbol{b}$ has a non-negative solution if $\boldsymbol{b} \geq \boldsymbol{0}$.*

Proof: From Theorem 3.2, there exists $(A^T A)^{-1} \geq 0$. Thus, the non-negative solution follows. □

Let $\overline{A}$ be a class of square matrices with positive diagonal and non-positive off-diagonal elements. A matrix $A \in \overline{A}$ is called an M-matrix if $A^{-1} \geq 0$. Thus, an M-matrix is monotone and will give a non-negative solution as in Lemma 3.4. More details on properties of the monotone matrix and M-matrix can be found in [1, 4, 7].

If columns of A form an NIP set, then we have a nice condition to obtain the positive solution.

Theorem 3.5 *Let A be an $n \times n$ matrix. Suppose that columns of the matrix $[A \,|\, -\boldsymbol{b}]$ form an NIP set, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has a positive solution.*

Proof: Let A_i for $i = 1, 2, \ldots, n$ denote columns of A, and $A' = \{A_i \,|\, i = 1, 2, \ldots, n\}$ From Corollary 2, A' is independent. Therefore A is non-singular and the system $A\boldsymbol{x} = -\boldsymbol{b}$ has a solution. Thus, there exists y_i for $i = 1, 2, \ldots, n$ such that $-\boldsymbol{b} = \sum\limits_{i=1}^{n} A_i y_i$. Since A' is an NIP set and $\langle -\boldsymbol{b}, A_i \rangle \leq 0$ for all $i = 1, 2, \ldots, n$, then, by Theorem 2.5, we have that $y_i < 0$ for all i. Thus the system $A\boldsymbol{x} = \boldsymbol{b}$ has the solution $\boldsymbol{x} = -\boldsymbol{y} > \boldsymbol{0}$. □

4. Constructing the NIP set

We may sometimes want to construct an NIP set in a linear space. The following procedure can produce an NIP set in the given space.

Let S be a subset of a vector space. The procedure to construct an NIP set in the space Lin S will be as the following:

1. Construct an orthonormal basis $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ for Lin S.

2. Let $\alpha_1 = \beta_1$. For $j = 2, \ldots, n$, let

$$\alpha_j = \beta_j - \frac{1}{j.n} \sum_{i=1}^{j-1} \beta_i,$$

and
$$\alpha_{n+1} = -\sum_{i=1}^{n} \beta_i.$$

Theorem 4.1 *The set $S' = \{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ obtained from the above procedure is an NIP set.*

Proof: It is clear that Lin $S' =$ Lin S. We need to show that $\langle \alpha_j, \alpha_k \rangle < 0$ for all $j \neq k$.

Considure
$$\langle \alpha_j, \beta_k \rangle = \left\langle \beta_j - \frac{1}{j.n} \sum_{i=1}^{j-1} \beta_i, \beta_k \right\rangle$$

$$= \langle \beta_j, \beta_k \rangle - \frac{1}{j.n} \sum_{i=1}^{j-1} \langle \beta_i, \beta_k \rangle$$
$$= \begin{cases} 0, & \text{if } j < k \\ 1, & \text{if } j = k \\ -\frac{1}{j.n}, & \text{if } j > k \end{cases}$$

For $k < j \leq n$, we have:

$$\begin{aligned} \langle \alpha_j, \alpha_k \rangle &= \left\langle \alpha_j, \beta_k - \frac{1}{k.n} \sum_{i=1}^{k-1} \beta_i \right\rangle \\ &= \langle \alpha_j, \beta_k \rangle - \frac{1}{k.n} \sum_{i=1}^{k-1} \langle \alpha_j, \beta_i \rangle \\ &= -\frac{1}{j.n} - \frac{1}{k.n} \sum_{i=1}^{k-1} \frac{-1}{j.n} \\ &= -\frac{1}{j.n} \left(1 - \frac{k-1}{k.n} \right) < 0. \end{aligned}$$

And
$$\langle \alpha_j, \alpha_{n+1} \rangle = \left\langle \alpha_j, - \sum_{i=1}^{n} \beta_i \right\rangle$$

$$\begin{aligned} &= -\sum_{i=1}^{j-1} \langle \alpha_j, \beta_i \rangle - \langle \alpha_j, \beta_j \rangle - \sum_{i=j+1}^{n} \langle \alpha_j, \beta_i \rangle \\ &= -\sum_{i=1}^{j-1} \frac{-1}{j.n} - 1 - \sum_{i=j+1}^{n} 0 \\ &= \frac{j-1}{j.n} - 1 < 0. \end{aligned}$$

Therefore S' is an NIP set and Lin S' = Lin S. $\square$

References

[1] Cegielski, A., *Obtuse cones and gram matrices with non-negative inverse.* [Online], Available: www.Elsevier.com/locate/laa. 2001.

[2] Bazaraa, M. and C. M. Shetty. *Nonlinear Programming : Theory and Algorithms.* John Wiley @ Sons. New York, 1979.

[3] Hoffman, K. and R. Kunze, *Linear Algebra*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

[4] Kiwiel, K. C., *Monotone Gram Matrices and Deepest Surrogate Inequalities in Accerated relaxation Methods for Convex Feasibility Problems*, North- Holland, Elsevier Science Inc., 1997.

[5] Lancaster, P. and M. Tismenetsky, *The Theory of Matrices; 2^{nd} Edition*, Academic Press, Orlando, 1985.

[6] Noble, B. and J. W. Daniel, *Applied Linear Algebra, 2^{nd} Edition*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1977.

[7] Poole, G and B. Thomas, *A Survey on M-matrices*. SIAM Review, (1974), **12**(4), 419-427.