East-West J. of Mathematics: Vol. 11, No 2 (2009) pp. 165-183

A GENERALIZATION OF THE ZARISKI TOPOLOGY OF ARBITRARY RINGS FOR MODULES

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Abstract

Let M be a left R-module. The set of all prime submodules of M is called the spectrum of M and denoted by $\mathrm{Spec}({}_RM)$, and that of all prime ideals of R is denoted by $\mathrm{Spec}(R)$. For each $\mathcal{P} \in \mathrm{Spec}(R)$, we define $\mathrm{Spec}_{\mathcal{P}}({}_RM) = \{P \in Spec({}_RM) : Ann_\ell(M/P) = \mathcal{P}\}$. If $\mathrm{Spec}_{\mathcal{P}}({}_RM) \neq \emptyset$, then $P_{\mathcal{P}} := \bigcap_{P\in Spec_{\mathcal{P}}({}_RM)} P$ is a prime submodule of M and $P \in \mathrm{Spec}_{\mathcal{P}}({}_RM)$. A prime submodule Q of M is called a lower prime submodule provided $Q = P_{\mathcal{P}}$ for some $\mathcal{P} \in \mathrm{Spec}(R)$. We write $\ell.\mathrm{Spec}({}_RM)$ for the set of all lower prime submodules of M and call it *lower spectrum* of M. In this article, we study the relationships among various module-theoretic properties of M and the topological conditions on $\ell.\mathrm{Spec}({}_RM)$ (with the Zariski topology). Also, we topologies $\ell.\mathrm{Spec}({}_RM)$ with the patch topology, and show that for every Noetherian left R-module M, $\ell.\mathrm{Spec}({}_RM)$ with the patch topology is a compact, Hausdorff, totally disconnected space. Finally, by applying Hochster's characterization of a spectral space, we show that if M is a Noetherian left R-module, then $\ell.\mathrm{Spec}({}_RM)$ with the Zariski topology is a spectral space, i.e., $\ell.\mathrm{Spec}({}_RM)$ is homeomorphic to $\mathrm{Spec}(S)$ for some commutative ring S. Also, as an application we show that for any ring R with ACC on ideals $\mathrm{Spec}(R)$ is a spectral space.

Key words: Prime submodule; Lower prime submodule; Prime spectrum; Zariski topology; Patch topology; Spectral space.
2000 AMS Mathematics Subject Classification: Primary 16Y60, 06B30; Secondary 16D80; 16S90.

0. Introduction.

Throughout, all rings are associative rings with identity elements, and all modules are unital left modules. The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. If N is a submodule (respectively proper submodule) of M we write $N \leq M$ (respectively $N \lneqq M$). We denote the left annihilator of a factor module M/N of M by $(N : M)$. We call M faithful if $(0 : M) = 0$.

Let Spec(R) denote the set of prime ideals of a ring R. The Zariski topology for Spec(R) is defined by letting $C \subseteq$Spec(R) be closed if and only if there exists an ideal I of R such that $C = \{\mathcal{P} \in Spec(R) \mid I \subseteq \mathcal{P}\}$ (see for example [1], [6] and [9]). A topological space is called spectral if it is homeomorphic to the prime spectrum of a commutative ring equipped with Zariski topology. Hachster [10], has characterized spectral spaces as follows:
A space X is spectral if and only if the following axioms hold:

(i) X is a T_0-space;
(ii) X is quasi-compact;
(iii) the quasi-compact open subsets of X are closed under finite intersection and form an open base;
(iv) each nonempty irreducible closed subset F of X has a generic point (i.e., F is the closure of a unique point).

Let M be a left R-module. A proper submodule P of M is called *prime* if $aRm \subseteq P$, for $a \in R$ and $m \in M$, implies $m \in P$ or $a \in (P : M)$ (see for example [7] and [20]). If M is a nonzero left R-module and $(0 : M) = (N : M)$ for all nonzero submodule N of M then M is called a prime module (see for example [9] and [21]). If M is a prime module, then $(0 : M) = \mathcal{P}$ is a prime ideal and it is called the *affiliated prime* of ${}_RM$. P is a prime submodule of M if and only if M/P is a prime module. Clearly, a two-sided ideal $\mathcal{P}$ of R is a prime ideal of R if and only if $\mathcal{P}$ is a prime submodule of ${}_RR$ (for more information about this and others related topics, see, for instance, [2], [3], [4], [5], [11], [13] and [16]). We define Spec(${}_RM$) for the set of all prime submodules of M and define $\text{Spec}_{\mathcal{P}}({}_RM) = \{P \in Spec({}_RM)$: the affiliated prime ideal of M/P is $\mathcal{P}\}$. Hence $\text{Spec}_{\mathcal{P}}({}_RM) = \{P \in Spec({}_RM) : (P : M) = \mathcal{P}\}$. If $\text{Spec}_{\mathcal{P}}({}_RM) \neq \emptyset$, then $P_{\mathcal{P}} := \bigcap_{P \in Spec_{\mathcal{P}}({}_RM)} P$ is a prime submodule of M and $P \in \text{Spec}_{\mathcal{P}}({}_RM)$ (see Proposition 1.10 in McCasland and Smith [20]). A prime submodule Q of M is called a lower prime submodule provided $Q = P_{\mathcal{P}}$ for some $\mathcal{P} \in$ Spec(R). Clearly, a left ideal $\mathcal{P}$ of any ring R is a lower prime submodule (left ideal) if and only if $\mathcal{P}$ is a prime two-sided ideal of R, and hence this notion of lower prime submodule is a natural generalization of the notion of prime two-sided ideal of rings to modules. We write ℓ.Spec(${}_RM$) for the set of all lower prime submodules of M and call it *lower spectrum* of M. Clearly, if $P, Q \in \ell$.Spec(${}_RM$), then $P = Q$ if and only if $(P : M) = (Q : M)$ and for any ring R we have ℓ.Spec(${}_RR$) =Spec(R). A module M over a commutative ring R is called a *multiplication module* if each submodule of M is of the form IM, where I is an ideal of R (see EL-Bast and Smith [8], for more details). It is clear that each prime submodule of a multiplication module M is of the form $\mathcal{P}M$ for some prime ideal $\mathcal{P}$ of R, and hence $\ell.\mathrm{Spec}({}_RM)=\mathrm{Spec}({}_RM)$. Also if R is a commutative ring, then for each R-module M and each prime ideal $\mathcal{P}$ of R such that $\mathcal{P}M \neq M$, $M(\mathcal{P}) := \{m \in M \ \mid Am \subseteq \mathcal{P}M \ \textit{for some ideal } A \nsubseteq \mathcal{P}\}$ is a submodule of M (see McCasland and Smith [20]). It is shown that for any free left R-module M, $\ell.Spec(M) = \{\mathcal{P}M \ : \ \mathcal{P} \in Spec(R),\ \mathcal{P}M \neq M\ \}$ and for any finitely generated faithful module M over a commutative ring R, $\ell.Spec(M) = \{M(\mathcal{P}) \ : \ \mathcal{P} \in Spec(R),\ \mathcal{P}M \neq M\ \}$ (see Proposition 1.3).

For any submodule N of a module M, define

$$V_\ell(N) = \{P_\mathcal{P} \in \ell.Spec(M) \mid \mathcal{P} \supseteq (N:M)\}.$$

Then:

(i) $V_\ell(0) = \ell.\mathrm{Spec}({}_RM)$ and $V_\ell(M) = \emptyset$,

(ii) $\bigcap_{i\in\Lambda} V_\ell(N_i) = V_\ell((\Sigma_{i\in\Lambda}(N_i:M))M)$ for any index set Λ

(iii) $V_\ell(N) \bigcup V_\ell(L) = V_\ell(N \bigcap L)$,

Also, for each submodule N of M we denote the complement of $V_\ell(N)$ in $\ell.\mathrm{Spec}({}_RM)$ by $U_\ell(N)$ (i.e., $U_\ell(N) = \{P_\mathcal{P} \in \ell.Spec(M) \mid \mathcal{P} \not\supseteq (N:M)\}$). From (i), (ii) and (iii) above, the family $\mathcal{T}_\ell(M) = \{U_\ell(N) | N \leq M\}$ is closed under finite intersections and arbitrary unions. Moreover, we have $U_\ell(M) = \ell.\mathrm{Spec}({}_RM)$ and $U_\ell(0) = \emptyset$. Therefore, $\mathcal{T}_\ell(M)$ is the family of open sets for a topology on $\ell.\mathrm{Spec}({}_RM)$ and call it the *lower Zariski topology* of M. This notion of lower Zariski topology of a module is analogous to that of the usual Zariski topology of a ring. In fact, for any ring R, the lower Zariski topology of ${}_RR$ and the usual Zariski topology of the ring R considered in [9], coincide. Also, the lower Zariski topology and the Zariski topology considered in [11], agree for multiplication modules (see also [11], [12], [17], [18] and [19]).

In this article, we study the relationships among various module-theoretic properties of M and the topological conditions on $\ell.\mathrm{Spec}({}_RM)$ (with the lower Zariski topology). Modules whose lower Zariski topology is T_1 are studied in Section 1. For example we show that for each R-module M, $\ell.\mathrm{Spec}({}_RM)$ is a T_1-space if and only if $Cl.K.dim(M) \leq 0$ (see [2], for the definition of classical Krull dimension of modules). This yields that if M is semisimple, R is a PI-ring and M is an Artinian R-module, or R is a commutative ring and M is co-semisimple, then $\ell.\mathrm{Spec}({}_RM)$ is a T_1-space.

In Section 2, we topologies $\ell.\mathrm{Spec}({}_RM)$ with the patch topology, and show that for every Noetherian left R-module M, $\ell.\mathrm{Spec}({}_RM)$ with the patch topology is a compact, Hausdorff, totally disconnected space. In the final section, comparing with $\mathrm{Spec}(R)$, R commutative, we investigate the lower Zariski topology of modules form the point of view of spectral spaces. In fact, by applying Hochster's characterization of a spectral space (see Hochster [10]), we show that if M is a Noetherian left R-module, then $\ell.\mathrm{Spec}({}_RM)$ with the Zariski topology is a spectral space, i.e., $\ell.\mathrm{Spec}(_RM)$ is homeomorphic to $\mathrm{Spec}(S)$ for some commutative ring S. As an application we conclude that for any ring R with ACC on ideals $\mathrm{Spec}(R)$ with the usual Zariski topology is a spectral space.

1. Some remarks about the lower Zariski topology of modules

Let X be a topological space and let x and y be points in X. We say that x and y can be separated if each lies in an open set which does not contain the other point. X is a T_1- space if any two distinct points in X can be separated. A topological space X is a T_1-space if and only if all points of X are closed in X (i.e., given any x in X, the singleton set $\{x\}$ is a closed set). (Note: for other terminology on a topological space not defined here we refer to Mankres [14].)

In the literature, there are two different generalizations of the classical Krull dimension for modules via prime dimension. In fact, the notion of prime dimension of a module M over a commutative ring R [denoted by $D(M)$ or $dim(M)$], was introduced by Marcelo and Masqué [15], as the maximum length of the chains of prime submodules of M (see [2] and [13]) for some known results about the prime dimension of modules). Also, the classical Krull dimension of rings has been extended to modules $_RM$ in [2], as the maximum length of the strong chains of prime submodules of M (allowing infinite ordinal values) and denoted by $Cl.K.dim(M)$ (see also [5] for another generalization of the classical Krull dimension of rings to modules). (Note: the chain $N_1 \subset_s N_2 \subset_s N_3 \subset_s \cdots$ of submodules of M is called a strong ascending chain if for each $i \in \mathbb{N}$, $N_i \subsetneqq N_{i+1}$ and also $(N_i : M) \subsetneqq (N_{i+1} : M)$; (see [2], for definition of the strong descending chain condition).

In the following result we give a characterization for the lower prime spectrum of finitely generated faithful modules over a commutative ring.

Theorem 1.1. *Let M be a left R-module. Then the following statements are equivalent:*

(1) *$\ell.Spec(_RM)$ with the lower Zariski topology is a T_1-space.*
(2) *$Cl.K.dim(M) \leq 0$.*

Proof. (1) $\Rightarrow$ (2). Assume that $\ell.Spec(M)$ with lower Zariski topology is a T_1-space. If $\ell.\mathrm{Spec}(_RM) = \emptyset$, then $Cl.K.dim(M) = -1$. Let $\ell.\mathrm{Spec}(_RM) \neq \emptyset$ and $P_1 \in \ell.\mathrm{Spec}(_RM)$. Then $\{P_1\}$ is a closed set in $\ell.\mathrm{Spec}(_RM)$. We claim that every prime submodule of M is a virtually maximal prime submodule, for if not, we assume that $P_1 \subset_s P_2$, where P_1, P_2 are lower prime submodules of M. Since $\{P_1\}$ is a closed set, $\{P_1\} = V_\ell(N)$, where $N \leq M$ and then $(N : M) \subseteq (P_1 : M)$. Thus we conclude that $P_1 \in V_\ell(N)$. Since $P_1 \subset_s P_2$, $(P_1 : M) \subset (P_2 : M)$. Therefore $P_2 \in V_\ell(N)$. Thus $P_2 \in \{P_1\}$, a contradiction. Thus every lower prime submodule of M is a virtually maximal prime submodule and then Cl.K.dim$(M) \leq 0$.

(2) $\Rightarrow$ (1). Suppose that Cl.K.dim$(M) \leq 0$. If Cl.K.dim$(M) = -1$, then ℓ.Spec$(_RM) = \emptyset$, i.e., ℓ.Spec$(_RM)$ is trivial space and so it is a T_1-space. Now let Cl.K.dim$(M) = 0$, i.e., ℓ.Spec$(_RM) \neq \emptyset$ and every prime submodule of M is a virtually maximal prime submodule. Thus for each lower prime submodule P of M, $V_\ell(P) = \{P\}$, and so $\{P\}$ is a closed set in ℓ.Spec$(_RM)$ i.e., ℓ.Spec$(_RM)$ is a T_1-space. □

The following corollary gives a wider class of modules M for which ℓ.Spec$(_RM)$ with the lower Zariski topology is a T_1-space.

Corollary 1.2. *Let M be a left R-module. Then:*

(a) *If M is semisimple, then ℓ.Spec$(_RM)$ with the lower Zariski topology is a T_1- space.*
(b) *If R is a PI-ring and M is Artinian, then ℓ.Spec$(_RM)$ with the lower zariski topology is a T_1-space.*
(c) *If R is commutative and M is co-semisimple, then ℓ.Spec$(_RM)$ with the lower zariski topology is a T_1- space.*

Proof. (a). By [5, Theorem 1.7] and Theorem 1.1.
(b). By [2, Theorem 1.10] and Theorem 1.1.
(c). By [2, Proposition 1.11] and Theorem 1.1. □

In the next proposition we give a characterization for ℓ.*Spec*(M) when M is a free module or R is a commutative ring and M is a finitely generated faithful R-module.

Proposition 1.3. *Let M be a nonzero left R-module. Then:*

(a) *If M is a free R-module, then for each prime ideal $\mathcal{P}$ of R, $\mathcal{P}M$ is a prime submodule of M such that $(\mathcal{P}M : M) = \mathcal{P}$. Moreover,*

$$\ell.Spec(_RM) = \{ \ \mathcal{P}M \mid for\ each\ ideal\ \mathcal{P}\ of\ R \ \}.$$

(b) *If R is commutative and M is a finitely generated faithful R-module, then*

$$\ell.Spec(M) = \{M(\mathcal{P}) \ : \ \mathcal{P} \in Spec(R), \ \mathcal{P}M \neq M \ \}.$$

Proof. (a). Since M is free R-module, then $M = \bigoplus R^{(I)}$, for index set I. One can easily see that for each prime ideal $\mathcal{P}$ of R, $\mathcal{P}M = \bigoplus \mathcal{P}^{(I)}$ is a prime submodule of M. On the other hand for each prime submodule P of M such that $(P : M) = \mathcal{P}$, $\mathcal{P}M \subseteq P$. Therefore

$$\mathcal{P}M \subseteq \bigcap_{P \in Spec_{\mathcal{P}}({}_RM)} P,$$

and hence,

$$\mathcal{P}M = \bigcap_{P \in Spec_{\mathcal{P}}({}_RM)} P.$$

(b). Let $\mathcal{P}$ be a prime ideal of R. By Lemma 2.1 in [5], $(M(\mathcal{P}) : M) = \mathcal{P}$. Since $\mathcal{P}M \subseteq M(\mathcal{P})$, $(\mathcal{P}M : M) = \mathcal{P}$. Thus by Proposition 1.8 in [20], $M(\mathcal{P})$ is a prime submodule of M. Then by Lemma 1.6 in [20], for each prime submodule K of M such that $(K : M) = \mathcal{P}$, $M(\mathcal{P}) \subseteq K$. Therefore

$$\ell.Spec({}_RM) = \{M(\mathcal{P}) \ : \ \mathcal{P} \in Spec(R), \ \mathcal{P}M \neq M \ \}. \ \square$$

We need the following evident lemma.

Lemma 1.4. *Let M be a left R-module. The for each submodule N of M, $V_\ell(N) = V_\ell(IM)$, where $I = (N : M)$. Consequently,*

$$\mathcal{T}_\ell(M) = \{U_\ell(IM) \mid I \text{ is an ideal of } R\}.$$

Let M be a left R-module and $\overline{R} = R/Ann(M)$. From the definition of the lower Zariski topology on ℓ.Spec$({}_RM)$, it is evident that the topological space $\ell.Spec(M)$ is closely related to Spec$(\overline{R})$, particularly, under the correspondence $\psi : \ell.Spec(M) \longrightarrow Spec(\overline{R})$ defined by $\psi(P) = (\overline{P : M})$ for every $P \in \ell$.Spec(M).

Proposition 1.5. *For any left R-module M, the natural map ψ is continuous map. More precisely, $\psi^{-1}(V(\overline{I})) = V_\ell(IM)$ for every ideal I of R containing $Ann(M)$.*

Proof. Suppose that $P \in \ell$.Spec(M) such that $P \in V_\ell(IM)$, then $I \subseteq (P : M)$ and so $(\overline{P : M}) \in V(\overline{I})$. Therefore $\psi(P) = (\overline{P : M}) \in V(\overline{I})$. Thus $P \in \psi^{-1}(V(\overline{I}))$. Conversely, if $P \in \psi^{-1}(V(\overline{I}))$, then $\psi(P) = (\overline{P : M}) \in V(\overline{I})$. Therefore $I \subseteq (P : M)$ and then $P \in V_\ell(IM)$. $\square$

Lemma 1.6. *For any left R-module M, the natural map $\psi : \ell.Spec(M) \longrightarrow Spec(\overline{R})$ is injective.*

Proof. Evident. $\square$

Lemma 1.7. *Let M be a left R-module. If the natural map ψ is surjective, then ψ is closed.*

Proof. By Proposition 1.5, ψ is a continuous map and $\psi^{-1}(V(\overline{I})) = V_\ell(IM)$, for each ideal I of R containing $Ann(M)$. Let $N \leq M$. Then $\psi^{-1}(V(\overline{(N : M)})) = V_\ell((N : M)M) = V_\ell(N)$. It follows that $\psi(V_\ell(N)) = \psi o \psi^{-1}(V(\overline{(N : M)})) = V(\overline{(N : M)})$ as ψ is bijective. □

Corollary 1.8. *Let M be a left R-module. If the natural map ψ is surjective, then ψ is homeomorphic.*

Proof. By Proposition 1.5, Lemma 1.6 and Lemma 1.7. □

Corollary 1.9. *Let M be a left R-module. Then the natural map ψ is homeomorphic in each of the following cases:*

(1) *M is a finitely generated nonzero module over commutative ring R;*
(2) *M is a faithfully flat nonzero module over commutative ring R;*
(3) *M is a free nonzero module over any ring R.*

Proof. (1) and (2) follow from [12, p. 3746, Theorem 2] and (3) follows from Proposition 1.3(a). □

2. Patch topologies associated to the lower spectrum of a module

We need to recall the patch topology (see [9] and [10], for definition and more details). Let X be a topological space. By the patch topology on X, we mean the topology which has as a sub-basis for its closed sets the closed sets and compact open sets of the original space. By a patch we mean a set closed in the patch topology. The patch topology associated to a spectral space is compact and Hausdorff (see Hachster [10]). Also, the patch topology associated to the Zariski topology of a ring R (not necessarily commutative) with ACC on ideals is compact and Hausdorff (see [9, Proposition 16.1]).

Definition 2.1. Let M be a left R-module, and let $\mathcal{V}_\ell(M)$ be the family of all subsets of $\ell.\mathrm{Spec}({}_RM)$ of the form $V_\ell(N)\bigcup U_\ell(K)$ where $V_\ell(N)$ is any lower Zariski-closed subset of $\ell.\mathrm{Spec}({}_RM)$ and $U_\ell(K)$ is a lower Zariski-quasi-compact subset of $\ell.\mathrm{Spec}({}_RM)$. Clearly $\mathcal{V}_\ell(M)$ is closed under finite unions and contains $\ell.\mathrm{Spec}({}_RM)$ and the empty set, since $\ell.\mathrm{Spec}({}_RM)$ equals $V_\ell(0)\bigcup U_\ell(0)$ and the empty set equals $V_\ell(M)\bigcup U_\ell(0)$. Therefore $\mathcal{V}_\ell(M)$ is basis for the family of closed sets of a topology on $\ell.\mathrm{Spec}({}_RM)$, and call it *lower patch topology* (or *lower constructible topology*) of M. Thus

$$\mathcal{V}_\ell(M)=\Big\{V_\ell(N)\bigcup U_\ell(K)\mid N,K\le M,\ U_\ell(K)\ \textit{is lower Zariski-quasi-compact}\Big\},$$

and hence we obtain the family

$$\mathcal{U}_\ell(M)=\Big\{V_\ell(K)\bigcap U_\ell(N)\mid N,K\le M,\ U_\ell(K)\ \textit{is lower Zariski-quasi-compact}\Big\},$$

which is a basis for the *open* sets of the lower patch topology, i.e., the patch-open subsets of $\ell.\mathrm{Spec}({}_RM)$ are precisely the unions of sets from $\mathcal{U}_\ell(M)$. We denote the patch-topology of $\ell.\mathrm{Spec}({}_RM)$ by $\mathcal{T}_{\ell p}(M)$.

We need the following definition (a slightly different notion of lower patch topology).

Definition 2.2. Let M be a left R-module, and let $\widetilde{\mathcal{U}}_\ell(M)$ be the family of all subsets of $\ell.\mathrm{Spec}({}_RM)$ of the form $V_\ell(N)\bigcap U_\ell(K)$ where $N,\ K\le M$. Clearly $\widetilde{\mathcal{U}}_\ell(M)$ contains $\ell.\mathrm{Spec}({}_RM)$ and the empty set, since $\ell.\mathrm{Spec}({}_RM)$ equals $V_\ell(0)\bigcap U_\ell(M)$ and the empty set equals $V_\ell(M)\bigcap U_\ell(0)$. Let $\widetilde{\mathcal{T}}_{\ell p}(M)$ to be the collection $\widetilde{U}$ of all unions of elements of $\widetilde{\mathcal{U}}_\ell(M)$. Then $\widetilde{\mathcal{T}}_{\ell p}(M)$ is a topology on $\ell.\mathrm{Spec}({}_RM)$ and it is called the *finer lower patch topology* or the *finer lower constructible topology* (in fact, $\widetilde{\mathcal{U}}_l(M)$ is a basis for the finer lower patch topology of M).

Lemma 2.3. *Let M be an R-module and $P\in\ell.Spec({}_RM)$. Then for each finer lower patch-neighborhood $\mathcal{U}_\ell$ of P, there exists a submodule L of M such that $(P:M)\subset(L:M)$ and $P\in V_\ell(P)\bigcap U_\ell(L)\subseteq\mathcal{U}_\ell$.*

Proof . Since $P\in\mathcal{U}_\ell$, there exists a neighborhood of the form $V_\ell(K)\bigcap U_\ell(N)\subseteq\mathcal{U}_\ell$ such that $P\in V_\ell(K)\bigcap U_\ell(N)$ where $(P:M)\supseteq(K:M)$ and $(P:M)\not\supseteq(N:M)$. Since $P\in V_\ell(P)$ and $V_\ell(P)\subseteq V_\ell(K)$, we may replace $V_\ell(K)$ by $V_\ell(P)$. Now we claim that $V_\ell(P)\bigcap U_\ell(N)=V_\ell(P)\bigcap U_\ell((I+\mathcal{P})M)$, where $\mathcal{P}=(P:M)$ and $I=(N:M)$. Since $U_\ell(IM)\subseteq U_\ell((I+\mathcal{P})M)$,

$$V_\ell(P)\bigcap U_\ell(N)=V_\ell(P)\bigcap U_\ell(IM)\subseteq V_\ell(P)\bigcap U_\ell((I+\mathcal{P})M).$$

Suppose that $Q\in V_\ell(P)\bigcap U_\ell((I+\mathcal{P})M)$, then $Q\notin U_\ell(P)$. On the other hand $Q\in U_\ell((I+\mathcal{P})M)=U_\ell(N)\bigcup U_\ell(P)$. This follows that $Q\in U_\ell(N)$. Thus $V_\ell(P)\bigcap U_\ell(N)=V_\ell(P)\bigcap U_\ell((I+\mathcal{P})M)$. Now let $L=(I+\mathcal{P})M$. Then $\mathcal{P}\subset I+\mathcal{P}\subseteq(L:M)$ and $P\in V_\ell(P)\bigcap U_\ell(L)\subseteq\mathcal{U}_\ell$. □

Let X be a topological space. Then for each subset Y of $\ell.\mathrm{Spec}(M)$, we will denote the closure of Y in $\ell.\mathrm{Spec}(M)$ with finer lower patch topology by $\overline{Y}$.

Proposition 2.4. *Let M be an R-module and $Y \subseteq \ell.Spec(_RM)$. If $Q \in \overline{Y}$ with finer lower patch topology, then there exists $\mathcal{A} \subseteq Y$ such that $V_\ell(Q) = V_\ell(\bigcap_{P\in\mathcal{A}} P)$.*

Proof. Let $Q \in \overline{Y}$. If $Q \in Y$, then we are thorough. Thus we can assume that $Q \notin Y$. Let $\mathcal{A} = \{P \in Y \mid (Q : M) \subset (P : M)\}$. Since $Q \in U_\ell(M) \bigcap V_\ell(Q)$, there exists $P' \in Y$ such that $P' \in U_\ell(M) \bigcap V_\ell(Q)$. Since $Q \notin Y$, $(Q : M) \subset (P' : M)$ and hence $\mathcal{A} \neq \emptyset$. Since $(Q : M) \subseteq (P : M)$ for each $P \in \mathcal{A}$,

$$(Q : M) \subseteq \bigcap_{P\in\mathcal{A}} (P : M) = (\bigcap_{P\in\mathcal{A}} P : M).$$

If $\bigcap_{P\in\mathcal{A}} (P : M) \nsubseteq (Q : M)$, then

$$Q \in U_\ell(\bigcap_{P\in\mathcal{A}} P) \bigcap V_\ell(Q).$$

Since $Q \in \overline{Y}$, there exists $P'' \in Y$ such that

$$P'' \in U_\ell(\bigcap_{P\in\mathcal{A}} P) \bigcap V_\ell(Q).$$

Therefore $P'' \in V_\ell(Q)$ and hence $P'' \in \mathcal{A}$. But

$$(\bigcap_{P\in\mathcal{A}} P : M) = \bigcap_{P\in\mathcal{A}} (P : M) \subseteq (P'' : M).$$

Thus $P'' \notin U_\ell(\bigcap_{P\in\mathcal{A}} P)$. a contradiction. Thus $\bigcap_{P\in\mathcal{A}} (P : M) \subseteq (Q : M)$, and hence

$$V_\ell(Q) = V_\ell(\bigcap_{P\in\mathcal{A}} P) = V_\ell(\bigcap_{P\in\mathcal{A}} (P : M)M). \ \square$$

Proposition 2.5. *Let M be a left R-module. Then $\ell.Spec(M)$ with the finer lower patch topology is Hausdorff. Moreover, $\ell.Spec(_RM)$ with this topology is totally disconnected.*

Proof. Suppose that P, $Q \in \ell.Spec(M)$ are distinct points. Since $P \neq Q$, $(P : M) \neq (Q : M)$. Therefore either $(P : M) \nsubseteq (Q : M)$ or $(Q : M) \nsubseteq (P : M)$. Assume that $(P : M) \nsubseteq (Q : M)$. By Definition 2.2, $U_1 := U_\ell(M) \bigcap V_\ell(P)$ is a finer lower patch-neighborhood of P and since $(P : M) \nsubseteq (Q : M)$, $U_2 := U_\ell(P) \bigcap V_\ell(Q)$ is a finer lower patch-neighborhood of Q. Clearly $U_\ell(P) \bigcap V_\ell(P) = \emptyset$ and hence $U_1 \cap U_2 = \emptyset$. Thus $\ell.\text{Spec}(_RM)$ is a Hausdorff space. On the other hand for every submodule N of M, observer that the sets $U_\ell(N)$ and $V_\ell(N)$ are open in finer lower patch topology, since $V_\ell(N) = U_\ell(M) \bigcap V_\ell(N)$ and $U_\ell(N) = U_\ell(N) \bigcap V_\ell(0)$. Since $U_\ell(N)$ and $V_\ell(N)$ are complement of each other, they are both finer lower both-closed as well. Therefore, the finer patch topology on $\ell.\mathrm{Spec}({}_RM)$ has a basis of open sets which are also closed, and hence $\ell.\mathrm{Spec}({}_RM)$ is totally disconnected in this topology. □

An R-module M will be called *weakly Noetherian* if, for every element a in R and element m in M, the submodule $RaRm$ is finitely generated (see [20]). For any ring R, every Noetherian module is weakly Noetherian. If R is a commutative ring, then every R-module is weakly Noetherian.

Definition 2.6. An R-module M is called p^*-module if for each prime ideal $\mathcal{P}$ of R such that $(\mathcal{P}M : M) = \mathcal{P}$, there exists a prime submodule P of M such that $(P : M) = \mathcal{P}$.

For example for each ring R, ${}_RR$ is a P^*-module. By Proposition 1.3 (a), every finitely generated faithful module over a commutative ring R is a P^*-module. Also every torsion free divisible module over any domain is a P^*-module. Now we show that every Noetherian left R-module M is also a P^*-module.

Lemma 2.7. *Let M be a Notherian left R-module. Then M is p^*-module.*

Proof. Let M be a Notherian left R-module. Then M is finitely generated and weakly Notherian. By [20, Proposition 1.8], for each prime ideal $\mathcal{P}$ of R, $M(\mathcal{P})$ is a prime submodule of M such that $(\mathcal{P}M : M) = \mathcal{P}$. □

Theorem 2.8. *Let R be a ring and M be a p^*-module such that $R/Ann(M)$ has ACC on ideals. Then $\ell.Spec({}_RM)$ with the finer lower patch topology is a compact space.*

Proof. Suppose M is a p^*-module such that $R/Ann(M)$ has ACC on ideals. Let $\mathcal{A}$ be a family of finer lower patch-open sets covering $\ell.\mathrm{Spec}({}_RM)$ and suppose that no finite subfamily of $\mathcal{A}$ covers $\ell.\mathrm{Spec}({}_RM)$. Let

$$S = \{L \mid L \text{ is an ideal of } R \text{ such that } Ann(M) \subseteq L \text{ and no finite subfamily of } \mathcal{A} \textit{ covers } V_\ell(LM)\}.$$

Since $V_\ell(Ann(M)M) = V_\ell(0) = \ell.\mathrm{Spec}(M)$, $S \neq \emptyset$. We may use the ACC on ideals of $R/Ann(M)$ to choose an ideal $\mathcal{Q}$ of R maximal with respect to the property that no finite subfamily of $\mathcal{A}$ covers $V_\ell(\mathcal{Q}M)$ (i.e., $\mathcal{Q}$ is a maximal element of S). It is clear that $\mathcal{Q}M \neq M$. We claim that $\mathcal{Q}$ is a prime ideal of R, for if not, suppose that I and J are two ideals of R properly containing

$\mathcal{Q}$ and $IJ \subseteq \mathcal{Q}$. Then $V_\ell(IM)$ and $V_\ell(JM)$ covered by finite subfamily of $\mathcal{A}$. Suppose $P \in V_\ell(IJM)$, then $IJ \subseteq \mathcal{P} := (P : M)$. Since $\mathcal{P}$ is prime, either $I \subseteq \mathcal{P}$ or $J \subseteq \mathcal{P}$, and hence either $P \in V_\ell(IM)$ or $P \in V_\ell(JM)$. Thus $V_\ell(IJM)$ covered by a finite subfamily of $\mathcal{A}$. Since $IJ \subset \mathcal{Q}$, then $V_\ell(\mathcal{Q}M) \subseteq V_\ell(IJM)$. Thus $V_\ell(\mathcal{Q}M)$ covered by finite subfamily of $\mathcal{A}$, a contradiction. Thus $\mathcal{Q}$ is a prime ideal of R. We claim that $(\mathcal{Q}M : M) = \mathcal{Q}$, for if not, then there exists an ideal $\mathcal{Q}_1$ of R such that $\mathcal{Q}_1 = (\mathcal{Q}M : M)$ and $\mathcal{Q} \subset \mathcal{Q}_1$. This follows that $\mathcal{Q}M = \mathcal{Q}_1 M$ and so no finite subfamily of $\mathcal{A}$ covers $V_\ell(\mathcal{Q}_1 M)$, contrary to maximality of $\mathcal{Q}$. Therefore $(\mathcal{Q}M : M) = \mathcal{Q}$ and since M is p^*-module, there exists $Q \in \ell.\mathrm{Spec}(M)$ such that $(Q : M) = \mathcal{Q}$. Let $U \in \mathcal{A}$ such that $Q \in U$. By Lemma 2.3, there exists a submodule K of M such that $\mathcal{Q} = (Q : M) \subset (K : M)$ and

$$Q \in U_\ell(K) \bigcap V_\ell(Q) \subseteq U.$$

Let $(K : M) = I$. By Lemma 1.4, we know that $U_\ell(K) = U_\ell(IM)$ and $V_\ell(Q) = V_\ell(\mathcal{Q}M)$, and so $Q \in U_\ell(IM) \bigcap V_\ell(\mathcal{Q}M) \subseteq U$. Since $\mathcal{Q} \subset I$, then $V_\ell(IM)$ can be covered by some finite subfamily $\mathcal{A}'$ of $\mathcal{A}$. But

$$V_\ell(\mathcal{Q}M) \setminus V_\ell(IM) = V_\ell(\mathcal{Q}M) \setminus [U_\ell(IM)]^c = V_\ell(\mathcal{Q}M) \bigcap U_\ell(IM) \subseteq U.$$

and so $V_\ell(\mathcal{Q}M)$ can be covered by $\mathcal{A}' \bigcup \{U\}$, contrary to our choice of Q. Thus there must exist a finite subfamily of $\mathcal{A}$ which covers $\ell.\mathrm{Spec}({}_RM)$. Therefore $\ell.\mathrm{Spec}({}_RM)$ is compact in the finer lower patch topology of M. □

It is well-known that if M is a Noetherian module over a commutative ring R, then $R/Ann(M)$ is a Noetherian ring. Thus by Lemma 2.7 and Theorem 2.8, we conclude that for each Noetherian module M over a commutative ring R, $\ell.\mathrm{Spec}({}_RM)$ with the finer lower patch topology is a compact space. Furthermore, by a similar method that used in the proof of Theorem 2.8, we show that this fact is also true for a Noetherian module over a non-commutative ring.

Theorem 2.9. *Let M be a Noetherian left R-module. Then $\ell.Spec({}_RM)$ with the finer lower patch topology is a compact space.*

Proof. Let M be a Noetherian left R-module, and let $\mathcal{A}$ be a family of finer lower patch-open sets covering $\ell.\mathrm{Spec}({}_RM)$. Suppose that no finite subfamily of $\mathcal{A}$ covers $\ell.\mathrm{Spec}({}_RM)$. Let

$$T = \{\mathcal{L}M \mid \mathcal{L} \text{ is an ideal of } R \text{ such that no finite subfamily of } \mathcal{A} \textit{ covers } V_\ell(\mathcal{L}M)\}.$$

Since $V_\ell(0M) = V_\ell(0) = \ell.\mathrm{Spec}(M)$, $T \neq \emptyset$. We may use the ACC on submodules of M to choose an ideal $\mathcal{F}$ of R such that $\mathcal{F}M$ maximal with respect to the property that no finite subfamily of $\mathcal{A}$ covers $V_\ell(\mathcal{F}M)$. Let $(\mathcal{F}M : M) = \mathcal{Q}$.

Then $V_\ell(\mathcal{F}M) = V_\ell(\mathcal{Q}M)$. It is clear that $\mathcal{Q}M \neq M$. We claim that $\mathcal{Q}$ is a prime ideal of R, for if not, suppose that I and J are two ideals of R properly containing $\mathcal{Q}$ and $IJ \subseteq \mathcal{Q}$. Then $\mathcal{F}M \subset IM$ and $\mathcal{F}M \subset JM$. Thus $V_\ell(IM)$ and $V_\ell(JM)$ covered by finite subfamily of $\mathcal{A}$. Suppose $P \in V_\ell(IJM)$, then $IJ \subseteq \mathcal{P} := (P : M)$. Since $\mathcal{P}$ is prime, either $I \subseteq \mathcal{P}$ or $J \subseteq \mathcal{P}$, and hence either $P \in V_\ell(IM)$ or $P \in V_\ell(JM)$. Thus $V_\ell(IJM)$ covered by a finite subfamily of $\mathcal{A}$. Since $IJ \subseteq \mathcal{P}$, then $V_\ell(\mathcal{Q}M) \subseteq V_\ell(IJM)$. Thus $V_\ell(\mathcal{Q}M)$ covered by finite subfamily of $\mathcal{A}$, a contradiction. Thus $\mathcal{Q}$ is a prime ideal of R. Now we claim that $(\mathcal{Q}M : M) = \mathcal{Q}$, for if not, then there exists an ideal $\mathcal{Q}_1$ of R such that $\mathcal{Q} \subset \mathcal{Q}_1$ and $\mathcal{Q}_1 = (\mathcal{Q}M : M)$. Therefore $\mathcal{Q}_1 M \subseteq \mathcal{Q}M \subseteq \mathcal{F}M$ and hence $\mathcal{Q}_1 \subseteq (\mathcal{F}M : M) = \mathcal{Q}$, a contradiction. Thus $(\mathcal{Q}M : M) = \mathcal{Q}$. By Lemma 2.7, M is p^*-module and so there exists $Q \in \ell.\mathrm{Spec}(M)$ such that $(Q : M) = \mathcal{Q}$. Let $U \in \mathcal{A}$ such that $Q \in U$. By Lemma 2.3, there exists a submodule K of M such that $\mathcal{Q} = (Q : M) \subset (K : M)$ and

$$Q \in U_\ell(K) \bigcap V_\ell(Q) \subseteq U.$$

Let $(K : M) = I$. By Lemma 1.4, we know that $U_\ell(K) = U_\ell(IM)$ and $V_\ell(Q) = V_\ell(\mathcal{Q}M)$, and so $Q \in U_\ell(IM) \bigcap V_\ell(\mathcal{Q}M) \subseteq U$. Since $\mathcal{Q} \subset I$, then $V_\ell(IM)$ can be covered by some finite subfamily $\mathcal{A}'$ of $\mathcal{A}$. But

$$V_\ell(\mathcal{Q}M) \setminus V_\ell(IM) = V_\ell(\mathcal{Q}M) \setminus [U_\ell(IM)]^c = V_\ell(\mathcal{Q}M) \bigcap U_\ell(IM) \subseteq U.$$

and so $V_\ell(\mathcal{Q}M)$ can be covered by $\mathcal{A}' \bigcup \{U\}$, contrary to our choice of Q. Thus there must exist a finite subfamily of $\mathcal{A}$ which covers $\ell.\mathrm{Spec}({}_RM)$. Therefore $\ell.\mathrm{Spec}({}_RM)$ is compact in the finer lower patch topology of M. □

We need the following evident lemma.

Lemma 2.10. *Assume that τ, τ^* are two topology on X such that $\tau \subseteq \tau^*$. If X is quasi-compact in τ^* then τ is also quasi-compact in τ.*

Theorem 2.11. *Let M be an R-module. If $\ell.Spec({}_RM)$ is compact with the finer lower patch topology, then for each submodule N of M, $U_\ell(N)$ is a quasi-compact subset of $\ell.Spec({}_RM)$ with the lower Zariski topology. Consequently, $\ell.Spec({}_RM)$ with the lower Zariski topology is quasi-compact.*

Proof. By Definition 2.2, for each submodule N of M, $V_\ell(N) = V_\ell(N) \bigcap U_\ell(M)$ is an open subset of $\ell.\mathrm{Spec}({}_RM)$ with finer lower patch topology, and hence, for each submodule N of M, $U_\ell(N)$ is a closed subset in $\ell.\mathrm{Spec}({}_RM)$ with finer lower patch topology. Since every closed subset of a compact space is compact, $U_\ell(N)$ is compact in $\ell.\mathrm{Spec}({}_RM)$ with finer lower patch topology and so by Lemma 2.10, it is quasi-compact in $\ell.\mathrm{Spec}({}_RM)$ with the lower Zariski topology. Now, since $\ell.\mathrm{Spec}({}_RM) = U_\ell(M)$, $\ell.\mathrm{Spec}({}_RM)$ is quasi-compact with lower Zariski topology.□

Corollary 2.12. *Let M be a left R-module. If ℓ.Spec$({}_RM)$ is compact with finer lower patch topology, then the finer lower patch topology and the lower patch topology of M coincide.*

Proof. By Theorem 2.11, for each submodule K of M, $U_\ell(K)$ is quasi-compact. Therefore for each $N, K \leq M$, $V_\ell(N) \bigcap U_\ell(K)$ is an element of the basis $\mathcal{U}_\ell(M)$ of the lower patch topology on ℓ.Spec$({}_RM)$. □

Corollary 2.13. *Let M be left R-module. If M is Noetherian or M is a p^*-module such that $R/Ann(M)$ has ACC on ideals, then the finer lower patch topology and the lower patch topology of M coincide.*

Proof. By Theorem 2.8, Theorem 2.9 and Corollary 2.12. □

We conclude this section with the following corollaries.

Corollary 2.14. *Let M be a p^*-module and $R/Ann(M)$ has ACC on ideals. Then ℓ.Spec$({}_RM)$ with the lower Zariski topology is a Hausdorf, compact, totally disconnected space.*

Proof. By Proposition 2.5, Theorem 2.8, and Corollary 2.13. □

Corollary 2.15. *Let M be a Notherian left R-module. Then ℓ.Spec$({}_RM)$ with the lower Zariski topology is a Hausdorf, compact, totally disconnected space.*

Proof. By Proposition 2.5, Theorem 2.9, and Corollary 2.13. □

3. Modules whose lower Zariski topologies are spectral

Let M be an R-module and let ℓ.*Spec*$({}_RM)$ be endowed with the lower Zariski topology. For each subset Y of ℓ.Spec$({}_RM)$, We will denote the closure of Y in ℓ.Spec$({}_RM)$ by $\overline{Y}$, and intersections of elements of Y by $\Im(Y)$ (note that if $Y = \emptyset$, then $\Im(Y) = M$).

A topological space X is called *irreducible* if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topology space X is called an *irreducible set* if the subspace Y of X is irreducible. For this to be so, it is necessary and sufficient that, for every pair of sets Y_1, Y_2 which are closed in X and satisfy $Y \subseteq Y_1 \cup Y_2$, $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (see [6, page 94]).

A topological space X is a T_0-space if and only if for any two distinct points in X there exists an open subset of X which contains one of the points but not the other. This characterization should be contrasted with an analogous characterization of T_1 spaces, where one can specify beforehand which points will belong to the open set.

We know that, for any ring R, Spec(R) is always a T_0-space for the usual Zariski topology. This is not true for $Spec({}_RM)$ (see [11, page 429]).

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a *generic point* of Y if $Y = \overline{\{y\}}$. Note that a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space.

Following Hochster [10], we say that a topological space X is a *spectral space* in case X is homeomorphic to Spec(S), with the Zariski topology, for some commutative ring S. Spectral spaces have been characterized by [10, page 52, Proposition 4] as the topological spaces X which satisfy the following conditions:

(i) X is a T_0-space;
(ii) X is quasi-compact;
(iii) the quasi-compact open subsets of X are closed under finite intersection and form an open base;
(iv) each irreducible closed subset of X has a generic point.

For any commutative ring R, Spec(R) is well-known to satisfy these condition (see [6, Chap II, 401-403]).

Corollary 3.1. *Let M ba a module over a commutative ring R. For each following cases $\ell.Spec(M)$ with lower Zariski topology is a spectral space:*

(1) *M is a finitely generated nonzero R-module ;*
(2) *M is a faithfully flat nonzero R-module;*
(3) *M is a free module.*

Proof. By Corollary 1.11 is clear. □

In this section, we will show that if ℓ.Spec(${}_RM$) with the finer lower patch topology is quasi compact, then ℓ.Spec(${}_RM$) with the lower Zariski topology is a spectral space.

Proposition 3.2. *Let M be an left R-module and let $Y = \{P_1, P_2, ..., P_k\}$ be a finite subset of $\ell.Spec({}_RM)$ with lower Zariski topology. Then $\overline{Y} = V_\ell(P_1) \cup V_\ell(P_2) \cup .... \cup V_\ell(P_k)$.*

Proof. Clearly, $Y \subseteq V_\ell(P_1) \cup V_\ell(P_2) \cup .... \cup V_\ell(P_k)$. Suppose F be any closed subset of $\ell.Spec(M)$ such that $Y \subseteq F$. Thus $F = V_\ell(N)$, for submodule $N$of M. Let $Q \in V_\ell(P_1) \cup V_\ell(P_2) \cup .... \cup V_\ell(P_k)$. Then there exists j $(1 \leq j \leq k)$ such that $Q \in V_\ell(P_j)$ and so $(P_j : M) \subseteq (Q : M)$. Since $P_j \in F$, $(N : M) \subseteq (P_j : M) \subseteq (Q : M)$, and hence $Q \in F$. Thus $V_\ell(P_1) \cup V_\ell(P_2) \cup .... \cup V_\ell(P_k) \subseteq F$. Therefore, $\overline{Y} = V_\ell(P_1) \cup V_\ell(P_2) \cup .... \cup V_\ell(P_k)$.□

The above proposition immediately yields that the following interesting result.

Corollary 3.3. *Let M be a left R-module. Then*

(a) $\{\overline{P}\} = V_\ell(P)$, *for all* $P \in \ell.Spec(M)$.
(b) $Q \in \{\overline{P}\}$ *if and only if* $(P : M) \subseteq (Q : M)$ *if and only if* $V_\ell(Q) \subseteq V_\ell(P)$.
(c) *The set* $\{P\}$ *is closed in* $\ell.Spec({}_RM)$ *if and only if* P *is a virtually maximal prime submodule of* M.

Proof. By Proposition 3.2 is clear. □

Lemma 3.4. *Let M be a left R-module and $P, Q \in \ell.Spec({}_RM)$. If $V_\ell(P) = V_\ell(Q)$, then $P = Q$.*

Proof. If $V_\ell(P) \subseteq V_\ell(Q)$, then $P \in V_\ell(Q)$. Therefore $(Q : M) \subseteq (P : M)$. On the other hand $V_\ell(Q) \subseteq V_\ell(P)$, then $Q \in V_\ell(P)$ and then $(P : M) \subseteq (Q : M)$. Therefore $(P : M) = (Q : M)$ and hence $P = Q$. □

Proposition 3.5. *Let M be a left R-module. Then $\ell.Spec({}_RM)$ with the lower Zariski topology is a T_0-space.*

Proof. Let P_1, P_2 be two distinct prime submodules of M. Since $P_1 \neq P_2$, by Lemma 3.3, $V_\ell(P) \neq V_\ell(Q)$. Therefore either $P \notin V_\ell(Q)$ or $Q \notin V_\ell(P)$. We assume that $P \notin V_\ell(Q)$. Thus $P \in U_\ell(Q)$, but $Q \notin U_\ell(Q)$. This means that $\ell.Spec(M)$ is a T_0-space. □

Lemma 3.6. *Let M be a left R-module. Then for each $P \in Spec({}_RM)$, $V_\ell(P)$ is irreducible.*

Proof. Let $V_\ell(P) \subseteq Y_1 \bigcup Y_2$, where Y_1 and Y_2 are closed sets. Then there exist $N_1, N_2 \leq M$ such that $Y_1 = V_\ell(N_1)$ and $Y_2 = V_\ell(N_2)$. Suppose that Q is a lower prime submodule of M such that $(Q : M) = (P : M)$. Since $Q \in V_\ell(P)$, either $Q \in Y_1$ or $Q \in Y_2$. Without loss of generality we can assume that $Q \in Y_1 = V_\ell(N_1)$, then $(N_1 : M) \subseteq (Q : M) = (P : M)$. Therefore $V_\ell(P) \subseteq V_\ell(N_1) = Y_1$. Thus $V_\ell(P)$ is irreducible. □

Corollary 3.7. *Let M be a left R-module and P be a prime submodule of M. If $Q \in \ell.Spec({}_RM)$ such that $(Q : M) = (P : M)$. Then Q is a generic point* *for the irreducible closed subset $V_\ell(P)$ of $\ell.Spec(_RM)$.*

Proof. By Lemma 3.6, $V_\ell(P)$ is irreducible closed subset of $\ell.\mathrm{Spec}(_RM)$. On the other hand by Corollary 3.3, $\{\overline{Q}\} = V_\ell(Q) = V_\ell(P)$. Thus Q is a generic point of irreducible closed subset $V_\ell(P)$. □

Let M be a left R-module and $Y \subseteq \ell.\mathrm{Spec}(_RM)$. We will denote the intersection of all elements in Y by $\Im(Y)$ and closure of Y in $\ell.\mathrm{Spec}(_RM)$ with the lower Zariski topology by $\overline{Y}$.

Proposition 3.8. *Let M be a left R-module and $Y \subseteq \ell.Spec(_RM)$. Then $V_\ell(\Im(Y)) = \bar{Y}$. Hence, Y is closed if and only if $V_\ell(\Im(Y)) = Y$.*

Proof. Clearly, $Y \subseteq V_\ell(\Im(Y))$. Let $V_\ell(N)$ be a closed subset of $\ell.\mathrm{Spec}(_RM)$ containing Y. Then $(N : M) \subseteq (P : M)$ for every $P \in Y$ so that $(N : M) \subseteq (\Im(Y) : M)$. Hence for every $Q \in V_\ell(\Im(Y))$, $(N : M) \subseteq (\Im(Y) : M) \subseteq (Q : M)$. Therefore $V_\ell(\Im(Y)) \subseteq V_\ell(N)$. Thus $\overline{Y} = V_\ell(\Im(Y))$. □

Now we show that if $Y \subseteq \ell.\mathrm{Spec}(_RM)$ such that $\Im(Y)$ is a prime submodule of M, then Y is irreducible.

Proposition 3.9. *Let M be a left R-module and $Y \subseteq \ell.Spec(_RM)$. If $\Im(Y)$ is a prime submodule of M, then Y is irreducible.*

Proof. Suppose that $P := \Im(Y)$ is a prime submodule of M. By Proposition 3.8, $\bar{Y} = V_\ell(P)$. Now let $Y \subseteq Y_1 \cup Y_2$, where Y_1, Y_2 are closes sets. Thus $\bar{Y} \subseteq Y_1 \cup Y_2$. Since $V_\ell(P) \subseteq Y_1 \cup Y_2$ and by Lemma 3.6, $V_\ell(P)$ is irreducible, $V_\ell(P) \subseteq Y_1$ or $V_\ell(P) \subseteq Y_2$. This follows that either $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (since $Y \subseteq V_\ell(P)$). Thus Y is irreducible. □

Let R be a ring and M be a left R-module. For any ideal I of R, $\sqrt{I}$ will denote the *radical* of I, that is

$$\sqrt{I} = \bigcap\{\mathcal{P} : \mathcal{P} \text{ is a prime ideal of } R \text{ and } I \subseteq \mathcal{P}\}$$

Also, for a submodule N of M the *prime radical* $\sqrt{N}$ (or $\mathrm{rad}_M(N)$) is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule then $\sqrt{N}$ is defined to be M. In particular, for any module M, we define $rad_R(M) = \sqrt{(0)}$. This is called *prime radical* of M. Thus, if M has a prime submodule, then $rad_R(M)$ is equal to the intersection of all the prime submodules in M but, if M has no prime submodule, then $rad_R(M) = M$.

Corollary 3.10. *Let M be a left R-module and $N \leq M$. If $\sqrt{N}$ is a prime submodule, then the subset $V_\ell(N)$ of $\ell.Spec(_RM)$ is irreducible with the* *lower Zariski topology.. Consequently, If $rad_R(M)$ is a prime submodule, then $\ell.Spec(_RM)$ is irreducible.*

Proof. By Proposition 3.9. □

Proposition 3.11. *Let M be a left R-module. If $\ell.Spec(_RM)$ is quasi-compact with the finer lower patch topology, then every irreducible closed subset of $\ell.Spec(_RM)$ with the lower Zariski topology has a generic point.*

Proof. The first we show that $Y = \bigcup_{P\in Y} V_\ell(P)$. Clearly $Y \subseteq \bigcup_{P\in Y} V_\ell(P)$. By Corollary 3.3(a), for each $P \in Y$ we have $V_\ell(P) = \{\overline{P}\} \subseteq \overline{Y}$, and since $\overline{Y} = Y$, $\bigcup_{P\in Y} V_\ell(P) \subseteq Y$. By Definition 2.2, for each $P \in Y$, $V_\ell(P) = V_\ell(P) \bigcap U_\ell(M)$ is an open subset of $\ell.\mathrm{Spec}(_RM)$ with the finer lower patch topology. Since Y is a closed subset in $\ell.\mathrm{Spec}(_RM)$ with the finer lower patch topology and since every closed subset of a compact space is compact, Y is compact in $\ell.\mathrm{Spec}(_RM)$ with the finer lower patch topology. Thus there exists a finite subset Y' of Y such that $Y = \bigcup_{P\in Y'} V_\ell(P)$. Also since Y is irreducible $Y = V_\ell(P)$ for some $P \in Y$ and so Y has a generic point. □

Corollary 3.12. *Let M be a left R-module with $|\ell.Spec(_RM)| < \infty$. Then every irreducible closed subset of $\ell.Spec(_RM)$ with the lower Zariski topology has a generic point.*

Corollary 3.13. *Let M be a p^*-module over a ring R such that $R/Ann(M)$ has ACC on ideals. Then every irreducible closed subset of $\ell.Spec(_RM)$ with the lower Zariski topology has a generic point.*

Corollary 3.14. *Let M be a Notherian left R-module. Then every irreducible closed subset of $\ell.Spec(_RM)$ with the lower Zariski topology has a generic point.*

Theorem 3.15. *Let M be a left R-module. If $\ell.Spec(_RM)$ is quasi-compact with the finer lower patch topology. Then $\ell.Spec(_RM)$ with the lower Zariski topology is a spectral space.*

Proof. By Proposition 3.5, $\ell.\mathrm{Spec}(_RM)$ is a T_0-space and by Theorem 2.12, $\ell.\mathrm{Spec}(_RM)$ is quasi-compact and has a basis of quasi-compact open subsets. Also, by Theorem 2.12, the family of quasi-compact open subsets of $\ell.Spec(M)$ is closed under finite intersections. Finally, by Proposition 3.11, every irreducible closed subset of $\ell.\mathrm{Spec}(_RM)$ has a generic point. Thus by Hochster's characterization of a spectral space, $\ell.\mathrm{Spec}(_RM)$ is a spectral space. □

Corollary 3.16. *Let M be a left R-module such that $|\ell.Spec(_RM)| < \infty$. Then $\ell.Spec(_RM)$ with lower Zariski topology is a spectral space.*

Proof. By Theorem 3.15 is clear. □

Corollary 3.17. *Let M be a p^*-module over ring R such that $R/Ann(M)$ has ACC on ideals. Then $\ell.Spec(_RM)$ with the lower Zariski topology is a spectral space.*

Proof. By Theorem 2.8 and Theorem 3.15 is clear. □

We conclude this article with the following interesting results for Noetherian modules and for arbitrary rings, respectively.

Corollary 3.18. *Let M be a Noetherian left R-module. Then $\ell.Spec(_RM)$ with the lower Zariski topology is a spectral space.*

Proof. By Theorem 2.9. Corollary 2.15 and Theorem 3.15. □

Corollary 3.19. *Let R be a ring (not necessary commutative) with ACC on ideals. Then $Spec(R)$ with the usual Zariski topology is a spectral space i.e., $Spec(R)$ is homeomorphic to $Spec(S)$ for some commutative ring S.*

Proof. By Corollary 2.14 and Theorem 3.15. □

Acknowledgments.

The research of the first author was in part supported by a grant from IPM (No. 85130016). Also, this work was partially supported by IUT (CEAMA). The authors would like to thank the Institute for Studies in Theoretical Physics and Mathematics (IPM) and IUT (CEAMA) for their financial support.

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