East-West J. of Mathematics: Vol. 10, No 1 (2008) pp.83-102

An Interior Proximal Method for Solving Monotone Generalized Variational Inequalities

Pham Ngoc Anh

Department of Scientific Fundamentals
Posts and Telecommunications Institute of Technology, Hanoi, Vietnam
anhpn@ptit.edu.vn

Abstract

We present a new method for solving generalized variational inequalities on polyhedra. The method is based on an interior-quadratic term which replaces the usual quadratic term. This leads to an interior proximal type algorithm. We first solve a monotone generalized variational inequalities satisfying a certain Lipschitz condition. Next, we combine this technique with line search technique to obtain a convergent algorithm for monotone generalized variational inequalities without Lipschitz condition. Finally some preliminary computational results are given.

1 Introduction

Let C be a polyhedral set on the real Euclidean space $\mathbb{R}^n$ defined by

$$C := \{x \in \mathbb{R}^n : \ Ax \le b\},$$

where A is an $p \times n$ matrix, $b \in \mathbb{R}^p, p \ge n$. We suppose that the matrix A is of maximal rank, i. e., rank$A = n$ and int$C = \{x : \ Ax < b\}$ is nonempty. Let F be a continuous mapping from D into $\mathbb{R}^n$, and φ be a lower semicontinuous convex function from C into $\mathbb{R}$. We say that a point x^* is a solution of the following generalized variational inequality if it satisfies

$$\langle F(x^*), x - x^* \rangle + \varphi(x) - \varphi(x^*) \ge 0 \ \ \forall x \in C, \qquad (VIP)$$

Key words: Variational inequalities, interior-quadratic method, monotonicity, line search algorithm, interior point method.
2000 AMS Mathematics Subject Classification: 65K10, 90C25

where $\langle .,. \rangle$ denotes the standard dot product in $I\!R^n$. Throughout the paper, we assume that the mapping F is monotone over C.

This generalized variational inequality problem have many important applications in economics, nonlinear analysis and have been studied by many researchers (see [9, 13, 15, 17, 19, 22, 24]).

It is well-known that the interior-quadratic technique is a powerfull tool for analyzing and solving optimization problems (see [6, 23]). Recently this technique has been used to develop proximal iterative algorithm for variational inequalities (see [5, 6, 8]).

In our recent paper [1] we have used the logrithmic quadratic function for pseudomonotone equilibrium on $I\!R^n := \{x = (x_1, ..., x_n) \in I\!R^n : \; x_i \geq 0 \;\; \forall i = 1, ..., n\}$ and developed algorithms for solving them.

In this paper we extend our results in [1, 2, 3] to the generalized variational inequality problem (VIP). Namely, we first develop a convergent algorithm for (VIP) with F being monotone function satisfying a certain Lipschitz type condition on C by using the interior-quadratic function. Next, in order to avoid the Lipschitz condition we will combine the line search method and this function to obtain a convergent algorithm for solving the generalized variational inequality problem (VIP) with the monotone function F.

The remaining part of the paper is structured as follows. In Section 2, we present a convergent algorithm for monotone and Lipschitz generalized variational inequality problems. In Section 3, we modify the algorithm by combining a line search with the interior-quadratic function, which allows avoiding the Lipschitz condition. Section 4 deals with some preliminary results of the proposed method.

2 Preliminaries on the interior-quadratic function

First, let us recall the well known concepts of monotonicity that will be used in the sequel.

Definition 2.1 *Let C be a convex set in $I\!R^n$, and $F : C \to I\!R^n$. The function F is said to be*
(i) monotone on C if for each $x, y \in C$, we have

$$\langle F(x) - F(y), x - y \rangle \geq 0;$$

(ii) strongly monotone on C with constant $\beta > 0$ if for each $x, y \in C$, we have

$$\langle F(x) - F(y), x - y \rangle \geq \beta ||x - y||^2;$$

(iii) Lipschitz with constant $L > 0$ on C (shortly $L-$Lipschitz), if we have

$$||F(x) - F(y)|| \leq L||x - y|| \quad \forall x, y \in C.$$

Remark 2.2 *Let A be $p \times n$ matrix, rank$A = n, C := \{x \in I\!R^n : \ Ax \leq b\}$, and $F : C \to I\!R^n$ be $L-$Lipschitz on C. Then we have*

$$||F(x) - F(y)|| \leq \bar{L}||A(x - y)|| \quad \forall x, y \in C,$$

where $\bar{A} := (a_{ij})_{n\times n}$ is a submatrix of A such that rank $\bar{A}$=n and

$$||\bar{A}^{-1}|| = \sup_{||x||=1} ||\bar{A}^{-1}x||,$$

and $\bar{L} = L||\bar{A}^{-1}||$.
Indeed, from

$$||F(x) - F(y)|| \leq L||x - y|| \quad \forall x, y \in C,$$

and

$$||x - y|| = ||\bar{A}^{-1}(\bar{A}(x - y))|| \leq ||\bar{A}^{-1}|| \, ||\bar{A}(x - y)|| \quad \forall x, y \in I\!R^n,$$

it follows that

$$||F(x) - F(y)|| \leq L||\bar{A}^{-1}|| \, ||A(x - y)|| \quad \forall x, y \in C.$$

Note that when φ is differentiable on some open set containing C, then, since φ is lower semicontinuous proper convex, the variational inequality (VIP) is equivalent to the following one (see [11, 12]):

Find $x^* \in C$ such that

$$\langle F(x^*) + \bigtriangledown\varphi(x^*), x - x^*\rangle \geq 0 \quad \forall x \in C.$$

In special case $\varphi = 0$, problem (VIP) can be written by the following:

Find $x^* \in C$ such that

$$\langle F(x^*), x - x^*\rangle \geq 0 \quad \forall x \in C. \tag{VI}$$

It is well known that the problem (VI) can be formulated as finding the zero point of the operator $T(x) = \varphi(x) + N_C(x)$ where

$$N_C(x) = \begin{cases} \{y \in C : \ \langle y, z - x\rangle \leq 0, \ \forall z \in C\} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases} \tag{2.1}$$

A classical method to solve this problem is the proximal point algorithm (see [2, 21]), which starting with any point $x^0 \in C$ and $\lambda_k \geq \lambda > 0$, iteratively updates x^{k+1} conforming the following problem:

$$0 \in \lambda_k T(x) + \nabla_x h(x, x^k), \tag{2.2}$$

where
$$h(x, x^k) = \frac{1}{2}||x - x^k||^2.$$

Recently, Auslender et al. [7] have proposed a new type of proximal interior method for solving problem (VI) on $C = I\!R^n_+$ through replacing function $h(x, x^k)$ by $d_\phi(x, x^k)$ which is defined as
$$d_\phi(x, y) = \sum_{i=1}^{n} y_i^2 \phi(y_i^{-1} x_i),$$
where
$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu(t - \log t - 1) & \text{if } t > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.3}$$
with $\nu > \mu > 0$. The fundamental difference here is that the term d_ϕ is used to force the iteratives $\{x^{k+1}\}$ to stay in the interior of $I\!R^n_+$.

Applying this idea to problem (VIP), in this paper we use the following function
$$d(x, y) = \begin{cases} \frac{1}{2}||x - y||^2 + \mu \sum_{i=1}^n y_i^2 (\frac{x_i}{y_i} \log \frac{x_i}{y_i} - \frac{x_i}{y_i} + 1) & \text{if } x > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.4}$$
with $\mu \in (0, 1)$ and $y \in C$. Let a_i denotes the rows of the matrix A, and define the following quantities:
$$\begin{aligned} &l_i(x) = b_i - \langle a_i, x \rangle, \\ &l(x) = (l_1(x), l_2(x), ..., l_p(x)), \\ &D(x, y) = d(l(x), l(y)). \end{aligned}$$
We denote by $\nabla_1 D(x, y)$ the gradient of $f(., y)$ at x for every $y \in C$. It is easy to see that
$$\nabla_1 D(x, y) = -A^T \big(l(x) - l(y) + \mu X_y \log \frac{l(x)}{l(y)}\big), \tag{2.5}$$
where $X_y = diag\big(l_1(y), ..., l_p(y)\big)$ and $\log \frac{l(x)}{l(y)} = \big(\log \frac{l_1(x)}{l_1(y)}, ..., \log \frac{l_p(x)}{l_p(y)}\big)$.

Now we consider the following gap function:
$$g(x) = \min\{\langle F(x), y - x \rangle + \varphi(y) - \varphi(x) + D(y, x) : \ y \in C\}. \tag{2.6}$$
Since C is closed convex and the objective function are strongly convex, the mathematical programming problem (2.6) is always solvable for any $x \in C$. Let $h(x)$ denote the unique solution to problem(2.6). h is a marginal mapping onto C. Observe that when φ is a constant function and $D(x, y) = \frac{1}{2}||x - y||^2$, h concides and becomes the marginal mapping for the projection gap function introduced in Fukushima (1992) (see [16]). The following lemma characterizes the solutions to problem (VIP) by means of the mapping h.

Lemma 2.3 *Suppose that the generalized variational inequality problem (VIP) has a solution. Then a point x^* is a solution to problem (VIP) if and only if $x^* = h(x^*)$.*

Proof. Let x^* be a solution to problem (VIP) and $h(x^*)$ be the unique solution to problem (2.6). Then

$$\langle F(x^*), h(x^*) - x^*\rangle + \varphi(h(x^*)) - \varphi(x^*) \geq 0. \tag{2.7}$$

Since $h(x^*)$ is the solution to problem (2.6), there exists a $z^* \in \partial_\varphi(h(x^*))$ such that

$$\langle F(x^*) + \nabla_1 D(h(x^*), x^*) + z^*, y - h(x^*)\rangle \geq 0 \quad \forall y \in C. \tag{2.8}$$

Replacing $y = x^*$ in (2.8) we get

$$\langle F(x^*) + \nabla_1 D(h(x^*), x^*) + z^*, x^* - h(x^*)\rangle \geq 0 \quad \forall y \in C. \tag{2.9}$$

Adding two inequalities (2.7) and (2.9) we obtain

$$\langle \nabla_1 D(h(x^*), x^*), x^* - h(x^*)\rangle \geq \langle z^*, h(x^*) - x^*\rangle + \varphi(x^*) - \varphi(h(x^*)). \tag{2.10}$$

Since $z^* \in \partial_\varphi(h(x^*))$, we have

$$\langle z^*, x^* - h(x^*)\rangle \leq \varphi(x^*) - \varphi(h(x^*)).$$

Thus

$$\langle z^*, x^* - h(x^*)\rangle - \varphi(x^*) + \varphi(h(x^*)) \leq 0. \tag{2.11}$$

From inequalities (2.10) and (2.11), it follows that

$$\langle \nabla_1 D(h(x^*), x^*), x^* - h(x^*)\rangle \geq 0.$$

By strongly monotonicity of $\nabla_1 D(., x^*)$ and $\nabla_1 D(x^*, x^*) = 0$, we have $x^* = h(x^*)$.

Conversely, suppose now $h(x^*) = x^*$. Then, by (2.8) we have

$$\langle F(x^*) + z^*, y - h(x^*)\rangle \geq 0 \quad \forall y \in C.$$

Since $z^* \in \partial_\varphi(h(x^*))$,

$$\langle z^*, y - x^*\rangle \leq \varphi(y) - \varphi(x^*) \quad \forall y \in C.$$

Adding the last two inqualities we have

$$\langle F(x^*), y - x^*\rangle + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in C,$$

which means that x^* is solution to problem (VIP).

□

Lemma 2.3 shows that the solution of the generalized variational inequality (VIP) can be approximated by an itertive procedure $x^{k+1} = h(x^k), k = 0, 1, ...$ where $c > 0$, x^0 is any starting point in C and $h(x^k)$ is the unique solution of the strongly convex program

$$\min\{\langle F(x^k), y\rangle + \varphi(y) + D(y, x^k) : \; y \in C\}.$$

Howerer, generally, the sequence $\{x^k\}_{k\geq 0}$ does not converge to a solution of problem (VIP) (see [13, 18]). Our goal now is to construct iteratively a sequence such that it converges to a solution to problem (VIP).

Algorithm 2.4 Step 0. *Choose $x^0 \in C, k := 0$, a positive sequence $\{c_k\}$ such that $c_k \to c > 0$ as $k \to +\infty$.*
Step 1. *Solve the strongly convex program:*

$$\min\{\langle F(x^k), y - x^k\rangle + \varphi(y) + \frac{1}{c_k} D(y, x^k) : \; y \in C\} \tag{2.12}$$

to obtain the unique solution y^k.
If $y^k = x^k$, then terminate: x^k is a solution to problem (VIP).
Otherwise go to Step 2.
Step 2. *Find x^{k+1} which is the unique solution to the strongly convex program:*

$$\min\{\langle F(y^k), y - y^k\rangle + \varphi(y) + \frac{1}{c_k} D(y, x^k) : \; y \in C\}.$$

Step 3. *Set $k := k + 1$, and return to Step 1.*

In the next proposition, we justify the stopping criterion.

Proposition 2.5 *If $y^k = x^k$, then x^k is a solution to problem (VIP).*

Proof. If the algorithm terminates at Step1, then $y^k = x^k$. It means that x^k is the solution to problem (2.6). By Lemma 2.3 it is a solution to problem (VIP).

□

In order to prove the convergence of Algorithm 2.4, we give the following key property of the sequence $\{x^k\}_{k\geq 0}$ generated by the algorithm.

Lemma 2.6 *Suppose that the function $F : C \to I\!R^n \cup \{+\infty\}$ is monotone, $L-$Lipschitz on C, and φ is convex function on C. Then, if the algorithm does not terminate, then we have*

$$\begin{aligned}||A(x^{k+1} - x^*)||^2 \leq & ||A(x^k - x^*)||^2 - \frac{1 - 3\mu - c_k||\bar{A}^{-1}||^2}{1+\mu}||A(x^{k+1} - y^k)||^2 \\ & - \frac{1 - 5\mu - c_k\bar{L}^2}{1+\mu}||A(x^k - y^k)||^2,\end{aligned}$$

where x^* is any solution to problem (VIP).

Proof. Since y^k is the solution to problem (2.12), from an optimization results in convex programming [20], we have

$$0 = F(x^k) + w_1 + \frac{1}{c_k}\nabla_1 D(y^k, x^k),$$

where $w_1 \in \partial\varphi(y^k)$. It follows that

$$\frac{1}{c_k}\langle \nabla_1 D(y^k, x^k), y - y^k\rangle = -\langle F(x^k) + w_1, y - y^k\rangle \quad \forall y \in C. \tag{2.13}$$

Since $w_1 \in \partial\varphi(y^k)$, we have

$$\varphi(y) - \varphi(y^k) \geq \langle w_1, y - y^k\rangle \quad \forall y \in C.$$

From (2.13) and this inequality it follows that

$$\frac{1}{c_k}\langle \nabla_1 D(y^k, x^k), y - y^k\rangle \geq \langle F(x^k), y^k - y\rangle + \varphi(y^k) - \varphi(y) \quad \forall y \in C. \tag{2.14}$$

Replacing y by x^*, we obtain

$$\frac{1}{c_k}\langle \nabla_1 D(y^k, x^k), x^* - y^k\rangle \geq \langle F(x^k), y^k - x^*\rangle + \varphi(y^k) - \varphi(x^*). \tag{2.15}$$

Note that x^* is a solution to probem (VIP),

$$\langle F(x^*), x - x^*\rangle + \varphi(x) - \varphi(x^*) \geq 0 \quad \forall x \in C.$$

By mononicity of F, it follows that

$$\langle F(x^k), x^k - x^*\rangle + \varphi(x^k) - \varphi(x^*) \geq 0. \tag{2.16}$$

Combinating (2.15) and (2.16) we obtain that

$$\frac{1}{c_k}\langle \nabla_1 D(y^k, x^k), x^* - y^k\rangle \geq \langle F(y^k), y^k - x^k\rangle + \varphi(y^k) - \varphi(x^k). \tag{2.17}$$

On the other hand, since x^{k+1} is the solution to the strongly convex program

$$\min\{\langle F(y^k), y\rangle + \varphi(y) + \frac{1}{c_k}D(y, x^k) : \ y \in C\},$$

in the same way, we also have

$$\frac{1}{c_k}\langle \nabla_1 D(x^{k+1}, x^k), x^* - x^{k+1}\rangle \geq \langle F(y^k), x^{k+1} - y^k\rangle + \varphi(x^{k+1}) - \varphi(y^k). \tag{2.18}$$

Now, applying the Lipschitz condition of F and Remark 2.2 with $x = x^k, y = y^k$, we obtain

$$\begin{aligned}\langle F(x^k) - F(y^k), x^{k+1} - y^k\rangle &\leq ||F(x^k) - F(y^k)||\ ||x^{k+1} - y^k|| \\ &\leq \frac{1}{2}||F(y^k) - F(x^k)||^2 + \frac{1}{2}||x^{k+1} - y^k||^2 \\ &\leq \frac{\bar{L}^2}{2}||A(y^k - x^k)||^2 + \frac{||\bar{A}^{-1}||^2}{2}\ ||A(x^{k+1} - y^k)||^2.\end{aligned}$$

Hence,

$$\begin{aligned}\langle F(y^k), x^{k+1} - y^k\rangle \geq &\langle F(x^k), x^{k+1} - y^k\rangle - \frac{\bar{L}^2}{2}||A(y^k - x^k)||^2 \\ &- \frac{||\bar{A}^{-1}||^2}{2}\ ||A(x^{k+1} - y^k)||^2. \end{aligned} \tag{2.19}$$

From (2.5), (2.18) and (2.19), we have

$$\begin{aligned}&\langle A(x^{k+1} - x^k), A(x^* - x^{k+1})\rangle \geq \\ &\qquad \mu\langle X_{x^k} log\frac{l(x^{k+1})}{l(x^k)}, A(x^* - x^{k+1})\rangle + c_k\langle F(x^k), x^{k+1} - y^k\rangle \\ &\qquad + c_k\varphi(x^{k+1}) - c_k\varphi(y^k) - \frac{c_k\bar{L}^2}{2}||A(y^k - x^k)||^2 \\ &\qquad - \frac{c_k||\bar{A}^{-1}||^2}{2}\ ||A(x^{k+1} - y^k)||^2. \end{aligned} \tag{2.20}$$

If $y = x^{k+1}$, inequality (2.14) becomes

$$\begin{aligned}\langle F(x^k), x^{k+1} - y^k\rangle + \varphi(x^{k+1}) - \varphi(y^k) &\geq \frac{1}{c_k}\langle \nabla_1 D(y^k, x^k), y^k - x^{k+1}\rangle \\ &= \frac{1}{c_k}\langle A^T\big(l(x^k) - l(y^k) - \mu X_{x^k}\log\frac{l(y^k)}{l(x^k)}\big), y^k - x^{k+1}\rangle \\ &= \frac{1}{c_k}\langle A(y^k - x^k), A(y^k - x^{k+1}\rangle \\ &\quad - \frac{1}{c_k}\mu\langle X_{x^k}\log\frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1})\rangle. \end{aligned} \tag{2.21}$$

From (2.20) and (2.21), it follows that

$$\begin{aligned}&\langle A(x^{k+1} - x^k), A(x^* - x^{k+1})\rangle \geq \\ &\qquad \mu\langle X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}, A(x^* - x^{k+1})\rangle - \frac{c_k\bar{L}^2}{2}||A(y^k - x^k)||^2 \\ &\qquad + \langle A(y^k - x^k), A(y^k - x^{k+1})\rangle - \frac{c_k||\bar{A}^{-1}||^2}{2}||A(x^{k+1} - y^k)||^2. \\ &\qquad - \mu\langle X_{x^k}\log\frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1})\rangle. \end{aligned} \tag{2.22}$$

Substituting

$$||A(x^k - x^*)||^2 = ||A(x^k - x^{k+1})||^2 + ||A(x^{k+1} - x^*)||^2 + 2\langle A(x^{k+1} - x^k), A(x^* - x^{k+1}\rangle\rangle$$

into (2.22), we obtain the estimation

$$\begin{aligned}||A(x^k - x^*)||^2 \geq\ & ||A(x^k - x^{k+1})||^2 + ||A(x^{k+1} - x^*)||^2 - c_k\bar{L}^2||A(y^k - x^k)||^2 \\ & + 2\langle A(y^k - x^k), A(y^k - x^{k+1})\rangle + 2\mu\langle X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}, A(x^* - x^{k+1})\rangle \\ & - 2\mu\langle X_{x^k}\log\frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1})\rangle - c_k||\bar{A}^{-1}||^2||A(x^{k+1} - y^k)||^2.\end{aligned}$$

Combining this inequality with the following equality

$$||A(x^{k+1} - x^k)||^2 = ||A(x^{k+1} - y^k)||^2 + ||A(x^k - y^k)||^2 + 2\langle A(x^{k+1} - y^k), A(y^k - x^k)\rangle,$$

we have

$$\begin{aligned}||A(x^{k+1} - x^*)||^2 \leq\ & ||A(x^k - x^*)||^2 - ||A(x^{k+1} - y^k)||^2 - ||A(x^k - y^k)||^2 \\ & + c_k||\bar{A}^{-1}||^2||A(x^{k+1} - y^k)||^2 - 2\mu\langle X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}, A(x^* - x^{k+1})\rangle \\ & + 2\mu\langle X_{x^k}\log\frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1})\rangle + c_k\bar{L}^2||A(y^k - x^k)||^2. \qquad (2.23)\end{aligned}$$

For each $t > 0$ we have $1 - \frac{1}{t} \leq \log t \leq t - 1$, then we obtain after multiplication by $l_i(x^*) \geq 0$ for each $i = 1, ..., p$,

$$l_i(x^k)l_i(x^*)\log\frac{l_i(x^{k+1})}{l_i(x^k)} \leq l_i(x^*)\big(l_i(x^{k+1}) - l_i(x^k)\big), \qquad (2.24)$$

and after multiplication by $-l_i(x^{k+1}) \leq 0$ for each $i = 1, ..., p$,

$$\begin{aligned}-l_i(x^k)l_i(x^{k+1})\log\frac{l_i(x^{k+1})}{l_i(x^k)} &\leq -l_i(x^k)l_i(x^{k+1})\big(1 - \frac{l_i(x^k)}{l_i(x^{k+1})}\big) \\ &= l_i(x^k)\big(l_i(x^k) - l_i(x^{k+1})\big). \qquad (2.25)\end{aligned}$$

Adding two inequalities (2.24) and (2.25), we obtain

$$\begin{aligned}&= |l_i(x^k) - l_i(x^*)|^2 + |l_i(x^k) - l_i(x^{k+1})|^2 \\ &\quad - |l_i(x^{k+1}) - l_i(x^*)|^2 \quad \forall i = 1, ..., p.\end{aligned}$$

These inequalities deduce that

$$\begin{aligned}&2\langle X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}, A(x^{k+1} - x^*)\rangle \leq \\ &\qquad ||A(x^k - x^*)||^2 + ||A(x^k - x^{k+1})||^2 - ||A(x^{k+1} - x^*)||^2. \qquad (2.26)\end{aligned}$$

In the same way, we also have

$$2\langle X_{x^k}\log\frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1})\rangle \leq ||A(x^k - y^k)||^2 + ||A(x^k - x^{k+1})||^2 - ||A(y^k - x^{k+1})||^2. \tag{2.27}$$

Adding the inequalities (2.23), (2.26) and (2.27), we get

$$\begin{aligned}||A(x^{k+1} - x^*)||^2 \leq\ & ||A(x^k - x^*)||^2 - ||A(x^{k+1} - y^k)||^2 - ||A(x^k - y^k)||^2 \\ & + c_k\bar{L}^2||A(y^k - x^k)||^2 + c_k||\bar{A}^{-1}||^2||A(x^{k+1} - y^k)||^2 + \mu(||A(x^k - x^*)||^2 \\ & + ||A(x^k - x^{k+1})||^2 - ||A(x^{k+1} - x^*)||^2) + \mu(||A(y^k - x^k)||^2 \\ & + ||A(x^{k+1} - x^k)||^2 - ||A(x^{k+1} - y^k)||^2),\end{aligned}$$

and consequently

$$\begin{aligned}(1+\mu)||A(x^{k+1} - x^*)||^2 \leq\ & \\ & (1+\mu)||A(x^k - x^*)||^2 - (1 + \mu - c_k||\bar{A}^{-1}||^2)||A(x^{k+1} - y^k)||^2 \\ & - (1 - \mu - c_k\bar{L}^2)||A(x^k - y^k)||^2 + 2\mu||A(x^{k+1} - x^k)||^2.\end{aligned} \tag{2.28}$$

Applying the following inequality

$$||A(x^{k+1} - x^k)||^2 \leq 2||A(x^{k+1} - y^k)||^2 + 2||A(x^k - y^k)||^2$$

to the last term in the right hand side of (2.28), we obtain

$$\begin{aligned}(1+\mu)||A(x^{k+1} - x^*)||^2 \leq\ & \\ & (1+\mu)||A(x^k - x^*)||^2 - (1 - 3\mu - c_k||\bar{A}^{-1}||^2)||A(x^{k+1} - y^k)||^2 \\ & - (1 - 5\mu - c_k\bar{L}^2)||A(x^k - y^k)||^2,\end{aligned}$$

which proves this lemma. □

The following theorem establishes the convergence of the algorithm.

Theorem 2.7 *Suppose that the function F is monotone and $L-$Lipschitz on C, that φ is convex and lower semicontinuous on C. Then, if the algorithm does not terminate and*

$$0 < \epsilon, 0 < \mu < \min\{\frac{1 - \epsilon - c_k||\bar{A}^{-1}||^2}{3}, \frac{1 - \epsilon - c_k\bar{L}^2}{5}\},$$

then the sequence $\{x^k\}_{k\geq 0}$ converges to a solution to problem (VIP).

Proof. From

$$0 < \mu < \min\{\frac{1 - \epsilon - c_k||\bar{A}^{-1}||^2}{3}, \frac{1 - \epsilon - c_k\bar{L}^2}{5}\}$$

and $\epsilon > 0$, we have

$$1 - 3\mu - c_k||\bar{A}^{-1}||^2 > 0 \quad \text{and} \quad 1 - 5\mu - c_k\bar{L}^2 > 0 \quad \forall k = 0, 1, ...$$

Then, using Lemma 2.6 we obtain that

$$||A(x^{k+1} - x^*)||^2 \leq ||A(x^k - x^*)||^2 \quad \forall k = 0, 1, ... \tag{2.29}$$

It means that the sequence $\{||A(x^k - x^*)||\}_{k\geq 0}$ is nonincreasing. Since it is bounded below by 0, it must be convergent. Since A is of maximal rank the function $u \to ||u||_A := ||Au||$ is norm on $I\!R^n$ and it follows that the sequence $\{||x^k - x^*||\}_{k\geq 0}$ converges. Then the sequence $\{x^k\}_{k\geq 0}$ is bounded and it has a subsequence $\{x^{k_i}\}_{i\geq 0}$ such that $x^{k_i} \to \bar{x}$ as $i \to +\infty$. From Lemma 2.6, we get

$$\frac{1 - 5\mu - c_k\bar{L}^2}{1+\mu}||A(x^k - y^k)||^2 \leq ||A(x^k - x^*)||^2 - ||A(x^{k+1} - x^*)||^2 \quad \forall k = 0, 1, ...$$

Applying these inequalities iteratively, we obtain

$$\sum_{k=0}^{n} \frac{1 - 5\mu - c_k\bar{L}^2}{1+\mu}||A(x^k - y^k)||^2 \leq ||A(x^0 - x^*)||^2 - ||A(x^{n+1} - x^*)||^2 \quad \forall k \geq 0.$$

As the sequence $\{||A(x^{n+1} - x^*)||\}_{k\geq 0}$ is convergent, passing $n \to +\infty$ we have

$$\lim_{k\to+\infty} \frac{1 - 3\mu - c_k||\bar{A}^{-1}||^2}{1+\mu}||A(x^k - y^k)||^2 = 0.$$

Using this with the assumption $1 - 5\mu - c_k\bar{L}^2 > \epsilon > 0$, we get

$$\lim_{k\to+\infty} \epsilon||A(x^k - y^k)|| = 0,$$

which implies

$$\lim_{i\to+\infty} ||A(\bar{x} - y^{k_i})|| = 0.$$

It holds that

$$\lim_{i\to\infty} y^{k_i} = \bar{x}.$$

Recall that y^{k_i} is the solution of the problem

$$\min\{\langle F(x^{k_i}), y\rangle + \varphi(y) + \frac{1}{c_{k_i}}D(y, x^{k_i}) : \ y \in C\}.$$

Then

$$\langle F(x^{k_i}), y^{k_i}\rangle + \varphi(y^{k_i}) + \frac{1}{c_{k_i}}D(y^{k_i}, x^{k_i}) \leq \langle F(x^{k_i}), y\rangle + \varphi(y) + \frac{1}{c_{k_i}}D(y, x^{k_i}) \ \forall y \in C.$$

Using the continuity of F, upper semicontinuity of $D(y,.)$, passing to the limit as $i \to +\infty$ we obtain

$$\langle F(\bar{x}), y-\bar{x}\rangle + \varphi(y) - \varphi(\bar{x}) + \frac{1}{c}D(y,\bar{x}) \geq 0 \quad \forall y \in C.$$

Then, there exists a $\bar{w} \in \partial\varphi(\bar{x})$ such that

$$\langle F(\bar{x}) + \bar{w} + \frac{1}{c}\nabla_1 D(\bar{x},\bar{x}), y-\bar{x}\rangle \geq 0 \quad \forall y \in C.$$

As $\nabla_1 D(\bar{x},\bar{x}) = 0$, this reduces to

$$\langle F(\bar{x}) + \bar{w}, y-\bar{x}\rangle \geq 0 \quad \forall y \in C.$$

Combining this inequality with the convexity of φ,

$$\varphi(y) - \varphi(\bar{x}) \geq \langle \bar{w}, y-\bar{x}\rangle \quad \forall y \in C,$$

we obtain that

$$\langle F(\bar{x}), y-\bar{x}\rangle + \varphi(y) - \varphi(\bar{x}) \geq 0 \quad \forall y \in C.$$

So $\bar{x}$ is a solution to problem (VIP).

Replacing x^* by $\bar{x}$ in (2.29) fields

$$||A(x^{k+1}-\bar{x})|| \leq ||A(x^k-\bar{x})|| \quad \forall k = 0, 1, ...$$

which implies that the sequence $\{||A(x^k-\bar{x})||\}_{k\geq 0}$ is convergent. We then have that the sequence $\{||x^k-\bar{x}||\}_{k\geq 0}$ is convergent. By the above proof, the sequence $\{x^k\}_{k\geq 0}$ has a subsequense converging to $\bar{x}$, we deduce that the whole sequence $\{x^k\}_{k\geq 0}$ converges to the solution $\bar{x}$ of problem (VIP).

□

3 The interior-quadratic proximal linesearch method

Convergence of Algorithm 2.4 requires that the function F satisfies the Lipschitz condition on C. This condition depends on positive constant L and in cases, it is unknown or difficult to approximate. So in this section, in order to avoid this assumption, we combine the interior-quadratic function with line search technique. This technique has been used widely in descent method for solving variational inequalitie problem (VI) on $C := \mathbb{R}^n_+$ (see [13, 23]).

The algorithm then can be described as follows.

Algorithm 3.1 Step 0. *Take* $x^0 \in C, k := 0$ *and a sequence* $\gamma_k \in (0;2)$ $\forall k \geq 0$.

Step 1. *Find* y^k *which is the solution to the strongly convex program:*

$$\min\{\langle F(x^k), y\rangle + \varphi(y) + \frac{1}{c_k}D(y, x^k) : \quad y \in C\}. \tag{3.1}$$

If $y^k = x^k$, *then stop.*

Otherwise go to Step 2.

Step 2. *Find* $\lambda_k \in (0,1)$ *as the smallest number such that*

$$\langle F((1-\lambda_k)x^k+\lambda_k y^k), y^k-x^k\rangle+\varphi(y^k)-\varphi((1-\lambda_k)x^k+\lambda_k y^k)+\frac{1}{2c_k}D(y^k, x^k) \leq 0. \tag{3.2}$$

Set $z^k := (1-\lambda_k)x^k + \lambda_k y^k$, *choose* $g^k \in F(z^k) + \partial\varphi(z^k)$.

If $g^k = 0$, *then stop.*

Otherwide go to Step 3.

Step 3. *Set* $\delta_k := \gamma_k \frac{\lambda_k\left(\langle F(z^k), z^k-y^k\rangle+\varphi(z^k)-\varphi(y^k)\right)}{(1-\lambda_k)||g^k||^2}$ *and*

$$x^{k+1} = P_C(x^k - \delta_k g^k),$$

$k := k+1$ *and return to Step 1.*

Recall that $P_C(x)$ denotes the projection of x on C.

First we have to show that there always exists $\lambda_k \in (0,1)$ as the smallest number satisfies (3.2). We suppose on the contrary that for every $\lambda \in (0,1)$, we have

$$\langle F((1-\lambda_k)x^k + \lambda_k y^k), y^k\rangle + \varphi(y^k) - \varphi((1-\lambda_k)x^k + \lambda_k y^k) + \frac{1}{2c_k}D(y^k, x^k) > 0.$$

Passing to the limit in the above inequality (as $\lambda \to 0^+$), by the continuity of $F(y)$, we obtain

$$\langle F(x^k), y^k - x^k\rangle + \varphi(y^k) - \varphi(x^k) + \frac{1}{2c_k}D(y^k, x^k) \geq 0. \tag{3.3}$$

Since y^k is a solution to (3.1), it follows that

$$\langle F(x^k), y\rangle + \varphi(y) + \frac{1}{c_k}D(y, x^k) \geq \langle F(x^k), y^k\rangle + \varphi(y^k) + \frac{1}{c_k}D(y^k, x^k).$$

Replacing y by x^k in the above inequality, we have

$$0 \geq \langle F(x^k), y^k - x^k\rangle + \varphi(y^k) - \varphi(x^k) + \frac{1}{c_k}D(y^k, x^k). \tag{3.4}$$

Then from (3.3) and (3.4) it follows that $D(x^k, y^k) = 0$, i.e., $d\big(l(x^k), l(y^k)\big) = 0$. Since $l(x) = b - Ax$ and A is maximal rank, we obtain $x^k = y^k$. This contracdicts to $x^k \neq y^k$ in Step 1.

Remark 3.2 *The smallest number $\lambda_k \in (0,1)$ in Step 2 of Aglgorithm 3.1 can be replaced by the following: With $\beta \in (0,1)$, we find n as the smallest natural number such that*

$$\langle F(\beta^n x^k+(1-\beta^n)y^k), y^k-x^k\rangle+\varphi(y^k)-\varphi(\beta^n x^k+(1-\beta^n)y^k)+\frac{1}{2c_k}D(y^k,x^k) \leq 0.$$

then set $\lambda_k := 1-\beta^n$.

In the next proposition, we justify the stopping criterion.

Proposition 3.3 *If $y^k = x^k$ or $g^k = 0$, then x^k is a solution to problem (VIP).*

Proof. If the algorithm terminates at Step 1, then $y^k = x^k$. It means that x^k is the solution to problem (3.1). Then

$$\langle F(x^k), y\rangle + \varphi(y) + D(y,x^k) \geq \langle F(x^k), x^k\rangle + \varphi(x^k) + D(x^k,x^k) \quad \forall y \in C.$$

From $D(x^k,x^k) = 0$, this inequality follows that x^k is a solution to problem (VIP).

If the algorithm terminates at Step 2, then $g^k = 0$, that means $0 \in F(z^k) + \partial\varphi(z^k)$. Thus $0 = F(z^k) + w^k$, where $w^k \in \partial\varphi(z^k)$. Hence

$$\begin{aligned}\varphi(x) - \varphi(z^k) &\geq \langle w^k, x-z^k\rangle \\ &= -\langle F(z^k), x-z^k\rangle \quad \forall x \in C.\end{aligned}$$

So z^k is a solution to problem (VIP). □

In order to prove the convergence of Algorithm 3.1, we give the following key property of the sequence $\{x^k\}_{k\geq 0}$ generated by the algorithm.

Lemma 3.4 *Suppose that the function F is monotone on C and φ is convex on C. Then, if the algorithm does not terminate, then we have*

$$||x^{k+1}-x^*||^2 \leq ||x^k-x^*||^2 - \frac{(2-\gamma_k)\delta_k^2}{\gamma_k}||g^k||^2,$$

where x^* is any solution to problem (VIP).

Proof. We have

$$\begin{aligned}||x^{k+1}-x^*||^2 &= ||P_k\big(x^k-\delta_k g^k\big)-x^*||^2 \\ &\leq ||x^k-x^*-\delta_k g^k||^2 \\ &= ||x^k-x^*||^2 - 2\delta_k\langle g^k, x^k-x^*\rangle + (\delta_k||g^k||)^2. \qquad (3.5)\end{aligned}$$

Note that, since x^* is a solution to problem (VIP),

$$\langle F(x^*), y - x^*\rangle + \varphi(y) - \varphi(x^*) \geq 0 \quad \forall y \in C.$$

Then by monotonicity, it follows that

$$\langle F(z^k), z^k - x^*\rangle + \varphi(z^k) - \varphi(x^*) \geq 0.$$

Combining this with

$$\begin{aligned}\langle g^k, x^k - x^*\rangle &= \langle g^k, x^k - z^k\rangle + \langle g^k, z^k - x^*\rangle \\ &\geq \langle g^k, x^k - z^k\rangle + \varphi(z^k) - \varphi(x^*) - \langle F(z^k), x^* - z^k\rangle,\end{aligned}$$

we obtain

$$\begin{aligned}\langle g^k, x^k - x^*\rangle &\geq \langle g^k, x^k - z^k\rangle \\ &= \frac{\lambda_k}{1-\lambda_k}\langle g^k, z^k - y^k\rangle \\ &\geq \frac{\lambda_k}{1-\lambda_k}\big(\langle F(z^k), z^k - y^k\rangle + \varphi(z^k) - \varphi(y^k)\big) \\ &= \frac{\delta_k}{\gamma_k}||g_k||^2. \end{aligned} \tag{3.6}$$

From (3.2) it follows that $\langle F(z^k), y^k - z^k\rangle + \varphi(y^k) - \varphi(z^k) < 0$. Hence

$$\delta_k := \gamma_k \frac{\lambda_k\big(\langle F(z^k), z^k - y^k\rangle + \varphi(z^k) - \varphi(y^k)\big)}{(1-\lambda_k)||g^k||^2} > 0. \tag{3.7}$$

Then from (3.5), (3.6) and (3.7), we have

$$\begin{aligned}||x^{k+1} - x^*||^2 &\leq ||x^k - x^*||^2 - 2\frac{\delta_k^2}{\gamma_k}||g^k||^2 + (\delta_k||g^k||)^2 \\ &= ||x^k - x^*||^2 - \frac{2-\gamma_k}{\gamma_k}(\delta_k||g^k||)^2,\end{aligned}$$

which proves the above lemma.

□

Now we are in a position to consider a convergence of Algorithm 3.1 in a case which does not terminate.

Theorem 3.5 *Suppose that the sequences $\gamma_k \in (0,2), c_k \to \bar{c}$ as $k \to \infty$, and functions F, φ satisfies the following conditions:*
(a) $\liminf \gamma_k(2-\gamma_k) > 0$.
(b) f is monotone on C.
(c) φ is lower semicontinuous on C.
Then, if Algorithm 3.1 doesn't terminate at Step 1 or Step 2, then the sequence $\{x^k\}_{k\geq 0}$ converges to x^* which is a solution to problem (VIP).

Proof. From Lemma 3.4, we have

$$\sum_{k=0}^{n} \frac{2-\gamma_k}{\gamma_k}(\delta_k||g^k||)^2 \le \sum_{k=0}^{n}(||x^k - x^*||^2 - ||x^{k+1} - x^*||^2)$$
$$= ||x^0 - x^*||^2 - ||x^{n+1} - x^*||^2 \quad \forall n \ge 0.$$

On the other hand, also since Lemma 3.4 deduces that $\{||x^k - x^*||\}$ is a decreasing sequence and is lower bounded by $||x^0 - x^*||$, then it must converge. It means that

$$\sum_{k=0}^{\infty} \frac{2-\gamma_k}{\gamma_k}(\delta_k||g^k||)^2 < +\infty.$$

Hence

$$\lim_{k\to\infty} \frac{2-\gamma_k}{\gamma_k}(\delta_k||g^k||)^2 = 0,$$

which together with $\lim_{k\to\infty} \inf(2-\gamma_k)\gamma_k > 0$ implies

$$\lim_{k\to\infty} \frac{\lambda_k\big(\langle F(z^k), z^k - y^k\rangle + \varphi(z^k) - \varphi(y^k)\big)}{(1-\lambda_k)||g^k||} = 0.$$

From the convergence of $\{||x^k - x^*||\}_{k\ge 0}$, we have that the sequence $\{x^k\}_{k\ge 0}$ is bounded. Then by the maximum theorem [4], we can deduce that the sequence $\{g^k\}_{k\ge 0}$ is bounded too. Thus

$$\lim_{k\to\infty} \frac{\lambda_k\big(\langle F(z^k), z^k - y^k\rangle + \varphi(z^k) - \varphi(y^k)\big)}{1-\lambda_k} = 0. \tag{3.8}$$

According to the rule (3.2), it is easy to see that

$$\frac{1}{2c_k} D(y^k, x^k) \le -\langle F(z^k), y^k - x^k\rangle - \varphi(y^k) + \varphi(z^k). \tag{3.9}$$

We consider two cases:
Case 1: If $\lim_{k\to\infty} \sup \lambda_k > 0$, then there exists $\bar\lambda \in (0,1]$ such that $\lambda_k \ge \bar\lambda \; \forall k \ge 0$. From (3.8) and inequality (3.9), we have

$$\lim_{k\to\infty} D(y^k, x^k) = 0. \tag{3.10}$$

Since the sequence $\{x^k\}_{k\ge 0}$ is bounded, hence it has a subsequence $\{x^k : k \in M\}$ converging to a point $\bar x$. Using the limit (3.10) we see that the subsequence $\{y^k : k \in M\}$ also converges to $\bar x$. Note that y^k is a solution to problem (3.1), hence

$$\langle F(x^k), y\rangle + \varphi(y) + \frac{1}{c_k} D(y, x^k) \ge \langle F(x^k), y^k\rangle + \varphi(y^k) + \frac{1}{c_k} D(y^k, x^k) \quad \forall y \in C.$$

Passing to the limit as $k \to \infty$ and using the continuity of F, the lower semicontinuity of φ, we have

$$\langle F(\bar{x}), y\rangle + \varphi(y) + \frac{1}{c_k}D(y,\bar{x}) \geq \langle F(\bar{x}), \bar{x}\rangle + \varphi(\bar{x}) + \frac{1}{c_k}D(\bar{x},\bar{x}) \quad \forall y \in C.$$

By Lemma 2.3, $\bar{x}$ is a solution to problem (VIP), thus the proof of the theorem in this case is complete.

Case 2: If $\lim_{k\to\infty}\sup \lambda_k = 0$, then since $\{x^k\}$ is bounded, we have some subsequence $\{x^k : k \in M\}$ converging to some point $\bar{x}$ as $k \to \infty$. From Step 1 of Algorithm 3.1, by lower semicontinuity of $\langle F(x^k), .\rangle + \varphi(.) + \frac{1}{c_k}D(., x^k)$, the sequence $\{y^k\}_{k\geq 0}$ is bounded too (see [4]). Thus, by taking a subsequence, if necessary, we may assume that the subsequence $\{y^k : k \in M\}$ also converges to some point $\bar{y}$. From

$$\langle F(x^k), y\rangle + \varphi(y) + \frac{1}{c_k}D(y^k, x^k) \geq \langle F(x^k), y^k\rangle + \varphi(y^k) + \frac{1}{c_k}D(y^k, x^k) \quad \forall y \in C,$$

by the lower semicontinuity of F, D and φ, taking the limit as $k \to \infty$, we can write

$$\langle F(\bar{x}), y\rangle + \varphi(y) + \frac{1}{\bar{c}}D(y,\bar{x}) \geq \langle F(\bar{x}), \bar{y}\rangle + \varphi(\bar{y}) + \frac{1}{\bar{c}}D(\bar{y},\bar{x}) \quad \forall y \in C. \tag{3.11}$$

Substituting $y = \bar{x}$ we then have

$$0 \geq \langle F(\bar{x}), \bar{y}\rangle + \varphi(\bar{y}) - \varphi(\bar{x}) + \frac{1}{\bar{c}}D(\bar{y},\bar{x}). \tag{3.12}$$

On the other hand, by Step 2 in Algorithm 3.1, since $\lambda_k \in (0, 1)$ is the smallest number satisfying

$$\langle F((1-\lambda_k)x^k + \lambda_k y^k), y^k - x^k\rangle + \varphi(y^k) - \varphi((1-\lambda_k)x^k + \lambda_k y^k) + \frac{1}{2c_k}D(y^k, x^k) \leq 0.$$

We deduce that

$$\langle F((1-\frac{1}{2}\lambda_k)x^k + \frac{1}{2}\lambda_k y^k), y^k - x^k\rangle + \varphi(y^k) - \varphi((1-\frac{1}{2}\lambda_k)x^k + \frac{1}{2}\lambda_k y^k) + \frac{1}{2c_k}D(y^k, x^k) > 0.$$

Passing $k \to \infty, k \in M$ the above inequality and using $\lim_{k\to\infty}\sup \lambda_k = 0$, we obtain

$$\langle F(\bar{x}), \bar{y} - \bar{x}\rangle + \varphi(\bar{y}) - \varphi(\bar{x}) + \frac{1}{2\bar{c}}D(\bar{y},\bar{x}) \geq 0.$$

This together with (3.12) implies $D(\bar{x},\bar{y}) = 0$, hence $\bar{x} = \bar{y}$. Then replacing $\bar{y}$ in (3.11) by $\bar{x}$, we deduce that

$$\langle F(\bar{x}), y - \bar{x}\rangle + \varphi(y) - \varphi(\bar{x}) + \frac{1}{\bar{c}}D(y,\bar{x}) \geq 0 \quad \forall y \in C.$$

The proof is complete.

□

4 Numerical Results

The airm of this section is to illustrate the proposed algorithms on a class of generalized variational inequality (VIP) where

$$C := \mathbb{R}^n_+, \varphi = 0, \text{and } F(x) = D(x) + Mx + q,$$

with the components of the $D(x)$ are $D_j(x) = d_j * arctan(x_j)$ $\forall j \geq 1$, d_j is chosen randomly in $(0, 1)$. The matrix $M = A^T A$ with A is $n \times n$ matrix whose entries are randomly generalized in the interval $(-1, 3)$. It is given by A. Bnouhachem (see [9]). Under these assumptions, it can be prove that F is continuous and monotone, that F is Lipschitz with constant $L \leq 1 + ||M||$.

Note that in this case, the subproblem

$$y^k = \text{argmin}\{\langle F(x^k), y - x^k\rangle + \frac{1}{c_k} d(y, x^k) : \ y \in C\}$$

where d is defined by (2.4). It is written as

$$y^k = \text{argmin}\{\langle F(x^k), y - x^k\rangle + \frac{1}{2c_k}||y - x^k||^2 + \frac{\mu}{c_k}\sum_{i=1}^{n}(x_i^k)^2\Big(\frac{y_i}{x_i^k}\log\frac{y_i}{x_i^k} - \frac{y_i}{x_i^k} + 1\Big) : y \in C_+\},$$

where

$$C_+ := \{x \in R^n : \ x_i > 0 \ \ \forall i = 1, ..., n\}.$$

It is not difficult to see that if we denote $y^k = (y_1^k, ..., y_n^k)$ and $F(x) = \big(F_1(x), ..., F_n(x)\big)$ for all $x \in C$, then for every $i = 1, ..., n$, we have y_i^k is the unique solution to the strongly convex problem

$$\min\{\frac{1}{2}t^2 + \eta_{ki}\ t + \xi_{ki}\ t\log t : \ t \in (0, +\infty)\},$$

where

$$\eta_{ki} := c_k F_i(x^k) - x_i^k - \mu x_i^k \ \log x_i^k - \mu x_i^k, \ \xi_{ki} := \mu x_i^k \ \ \forall i = 1, ..., n.$$

In test we take the logarithmic parameter $\mu = 0.01$, $c_k = 0.01$ $\forall k \geq 1$ and the tolerance 10^{-7}. We obtained the following computational results.

Iter(k)	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k
0	1	1	1	1	1	1	1
1	1.0470	0.8957	1.0302	1.0030	0.9289	0.9641	0.9084
2	1.0886	0.8120	1.0654	1.0099	0.8549	0.9311	0.8199
3	1.1243	0.7436	1.1005	1.0155	0.7793	0.8975	0.7347
4	1.1528	0.6847	1.1296	1.0134	0.7022	0.8584	0.6540
5	1.1735	0.6296	1.1475	0.9976	0.6232	0.8095	0.5793
6	1.1864	0.5710	1.1496	0.9632	0.5422	0.7473	0.5112
7	1.1915	0.5067	1.1344	0.9088	0.4597	0.6706	0.4497
8	1.1887	0.4364	1.1014	0.8340	0.3763	0.5794	0.3946
9	1.1779	0.3601	1.0502	0.7383	0.2926	0.4733	0.3455
10	1.1590	0.2774	0.9801	0.6214	0.2095	0.3522	0.3020
11	1.1317	0.1882	0.8906	0.4829	0.1277	0.2159	0.2640

Table 1. Numerical results: Algorithm 2.4 with $n = 7$.

The approximate solution obtained after eleven iterations is

$$x^{10} = (1.1317, 0.1882, 0.8906, 0.4829, 0.1277, 0.2159, 0.2640)^T.$$

References

[1] Anh P. N., *A logarithmic quadratic regularization method for solving pseudomonotone equilibrium problems*, Accepted by Acta Mathematica Vietnamica, 2008.

[2] Anh P. N., and Muu L. D., *Coupling the Banach contraction mapping principle and the proximal point algorithm for solving monotone variational inequalities*, Acta Mathematica Vietnamica, 29 (2004) 119-133.

[3] Anh P. N., Muu L.D., Nguyen V. H., and Strodiot J. J., *On the Contraction and Nonexpensiveness Properties of the Marginal Mappings in Generalized Variational Inequalities Involving Co-coercive Operators*, In Generalized Convexity and Monotonicity, Edited by A. Eberhard, N. Hadjisavvas and D.T. Luc Springer , Chap. 5, (2005) 89-111.

[4] Aubin J.P., and Ekeland I., *Applied Nonlinear Analysis*, Wiley, New York (1984).

[5] Auslender A., and Teboulle M., *Interior projection-like methods for monotone variational inequalities*, Mathematical Programming, 104 (2005) 39-68.

[6] Auslender A., Teboulle M., and Bentiba S., *A logarithmic-quadratic proximal method for variational inequalities*, J. of Computational Optimization and Applications, 12 (1999) 31-40.

[7] Auslender A., Teboulle M., and Bentiba S., *Iterior proximal and multiplier methods based on second order homogeneous kernels*, Mathematical Operation Research, 24 (1999) 646-668.

[8] Bnouhachem A. *An LQP method for psedomonotone variational inequalities*, J. of Global Optimization, 36 (2006) 351-363.

[9] Blum E., and Oettli W., *From optimization and variational inequality to equilibrium problems*, The Mathematics Student, 63 (1994) 127-149.

[10] Cohen G., *Auxiliary problem principle and decomposition of optimization problems*, J. of Optimization Theory and Applications, 32 (1980) 277-305.

[11] Cohen G., *Auxiliary principle extended to variational inequalities*, J. of Optimization Theory and Applications, 59 (1988) 325-333.

[12] Daniele P., Giannessi F., and Maugeri A., *Equilibrium Problems and Variational Models* , Kluwer (2003).

[13] Facchinei F., and Pang J.S., *Finite-Dimensional Variational Inequalities and Complementary Problems*, Springer-Verlag, NewYork (2003).

[14] Fang S.C., Petersen P.L., *Genralized Variational Inequalities*, J. of Optimization Theory and Applications, 38 (1982) 363-383.

[15] Farouq N. El., *Pseudomonotone variational inequalities: convergence of auxiliary problem method*, J. of Optimization Theory and Applications, 111 (2001) 305- 325.

[16] Fukushima M., *Equivalent Differentiable Optimization Problems and Descent Methods for Asymmetric Variational Inequality Problems*, Mathematical Programming, 53 (1992) 99-110.

[17] Konnov I. V., *Combined Relaxation Methods for Variational Inequalities*, Springer-Verlag, Berlin (2000).

[18] Mangasarian O. L., and Solodov M.V., *A linearly convergent derivative-free descent method for strongly monotone complementarity problem*, Computational Optimization and Applications, 14 (1999) 5-16.

[19] Noor M. A., *Iterative schemes for quasimonotone mixed variational inequalities*, SIAM J. Optimization, 50 (2001) 29-44.

[20] Rockafellar R. T., *Convex Analysis*, Princeton University Press (1970).

[21] Rockafellar R. T., *Monotone operators and the proximal point algorithm*, SIAM J. Control Optimization, 14 (1976) 877-898.

[22] Tan N. X., *On the existence of solutions of quasi-variational inclusion problems*, J. of Optimization Theory and Applications, 123 (2007) 619-638.

[23] Waltz R. A, Morales J. L., Nocedal J., and Orban D., *An interior algorithm for nonlinear optimization that combines line search and trust region steps*, Mathematical Programming, 107 (2006) 391-408.

[24] Zhu D. L., and Marcotte. P., *Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities*, SIAM J. Optimization, 6 (1996) 714-726.