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THE LOWER SEMI-CONTINUITY OF TOTAL CURVATURE IN SPACES OF CURVATURE BOUNDED ABOVE

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Abstract

In metric spaces of curvature bounded above in the sense of Alexandrov, the notions of (pointwise) curvature and total curvature can be defined. Certain properties of them were verified true, whereas many are still left unchecked. These include the lower semi-continuity of total curvature. In other words, it has not been known whether the relation $\kappa(\gamma) \leq \liminf_{m\to\infty} \kappa(\gamma_m)$, where γ_m is a sequence of curves that converges to a curve γ, holds in a more general setting than that of the Euclidean space. We present here the validity of this statement in spaces of curvature bounded above.

1 Introduction

The notion of curvature for a curve is one of the central concepts of geometry. A curvature, intuitively, measures the amount that a curve deviates away from being straight. These amounts are determined by the change of the direction of a curve at its points. Pointwise curvature tells how fast the direction changes at a point and total curvature measures the accumulative change as the entire curve is passed through. In a smooth case in Euclidean space, the total curvature is the integral of its (unsigned) pointwise scalar curvature with respect to arclength. For a more general case, the total curvature was introduced, among others, by A. D. Alexandrov in 1946 [3]. An extensive development of

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the theory on this subject can be found in [5]. There, the total curvature was defined by considering a total turn (rotation) of a sequence of polysegments inscribed in and arbitrary close to it. Recently, this idea has been extended to CAT(0) spaces by S. B. Alexander and R. L. Bishop [2] and then to CAT(K) spaces by C. Maneesawarng and Y. Lenbury [11, 12]. Certain properties of total curvature in CAT(K) spaces which are analogue to those in Euclidean space were verified true and the estimate of the length of a curve through its total curvature was also studied there. A variation of total curvature for closed curves was studied by A.A. Sama-Ae [14]. The main purpose of this paper is to prove the following theorem, which is an extension of the lower semi-continuity of total curvature for curves in Euclidean space [5] to those in spaces of curvature bounded above in the sense of Alexandrov.

Theorem *If a sequence of curves γ_m converges to a curve γ in a metric space of curvature bounded above, then*

$$\kappa(\gamma) \quad \leq \quad \liminf_{m\to\infty} \kappa(\gamma_m).$$

2 Definitions and Preliminaries

A metric space X is called a *CAT(K) space* if any two points are joined by a minimizing geodesic (a curve realizing distance in X) and for any minimizing triangle with perimeter less than $2\pi/\sqrt{K}(= \infty$ if K $\leq 0)$, the distance between any two points on the triangle is no greater than the distance between corresponding points on its comparison triangle in the model space S_K (the 2-dimensional spherical, Euclidean or hyperbolic space of constant curvature K accordingly as $K > 0$, $K = 0$ or $K < 0$). A metric space *has curvature bounded above* by K if every of its points is contained in an open neighborhood which is a CAT(K) space itself.

In a CAT(K) space, the notion of angles between two geodesics having a common endpoint can be defined. Actually, an angle between two general curves with a common endpoint has carefully been studied by Alexandrov in [4]. In the following proposition, the symbol $\angle pqr$, where p, q and r are points in a metric space, denotes the angle between geodesics joining q to p and r.

Proposition 1. [5, p.18] *Let x_n, y_n and z_n be sequences of points in a CAT(K) space X. If $x_n \to x, y_n \to y$ and $z_n \to z$ with $y \in X$ and $x, z \neq y$, then for sufficiently large n the angles $\angle x_n y_n z_n$ and $\angle x_n y z_n$ are defined and*

$$\limsup_{n\to\infty} \angle x_n y_n z_n \leq \lim_{n\to\infty} \angle x_n y z_n = \angle xyz.$$

The following concept of total curvature for curves in metric spaces of curvature bounded above was generalized for all real values K by Maneesawarng and Lenbury in [11, 12]. Consider first a CAT(K) space X. A *polysegment* is a curve $\sigma : [a, b] \to X$ for which a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$ exists in such a way that for each $i \in \{1, 2, \ldots, k\}$, the arc $\sigma|_{[t_{i-1},t_i]}$ is a minimizing geodesic, called a *geodesic segment* or simply a *geodesic*. A polysegment σ is *inscribed* in a curve $\gamma : [a, b] \to X$ if there are a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$ and a parametrization of σ such that $\sigma(t_i) = \gamma(t_i)$, and $\sigma|_{[t_{i-1},t_i]}$ is a geodesic for all $i \in \{1, 2, \ldots, k\}$.

If σ is a polysegment with ordered vertices $p_0, p_1, \ldots, p_k$, i.e., the p_i's correspond to ascending values of the parameter of σ, then the *angle* $\widehat{p_i}$ of σ at an interior vertex p_i is the angle subtended by the two geodesic segments $[p_{i-1}, p_i]$ and $[p_i, p_{i+1}]$. The *total rotation* $\kappa^*(\sigma)$ of σ is then defined by the sum of *rotations* of σ,

$$\kappa^*(\sigma) = \sum_{i=1}^{k-1} (\pi - \widehat{p_i}).$$

Following terminology used in [5, 12] for each polysegment σ inscribed in a curve γ, the *modulus* of σ associated with γ is defined as

$$\mu_\gamma(\sigma) = \max_{1 \le i \le k} \operatorname{diam}(\gamma|_{[t_{i-1},t_i]}),$$

and the *mesh* of σ associated with γ as

$$\tilde{\mu}_\gamma(\sigma) = \max_{1 \le i \le k} \ell(\gamma|_{[t_{i-1},t_i]}),$$

where $a = t_0 < t_1 < \cdots < t_k = b$ is a partition of $[a, b]$ associated with σ as above and $\ell(\gamma|_{[t_{i-1},t_i]})$ is the length of $\gamma|_{[t_{i-1},t_i]}$ for all $i \in \{1, 2, \ldots, k\}$. The *total curvature* $\kappa(\gamma)$ of γ is defined by the limit supremum

$$\kappa(\gamma) = \limsup_{\mu_\gamma(\sigma) \to 0} \kappa^*(\sigma) = \lim_{\varepsilon \to 0^+} \sup_{\sigma \in \Sigma_\varepsilon(\gamma)} \kappa^*(\sigma),$$

where for each $\varepsilon > 0$, $\Sigma_\varepsilon(\gamma)$ is the set of polysegments σ inscribed in γ such that $\mu_\gamma(\sigma) < \varepsilon$. If γ is itself a polysegment in a CAT(K) space, it was shown in [12] that its total curvature and its total rotation coincide. That total curvature is additive, i.e., the total curvature of a curve γ is the sum of the total curvatures of its subarcs (whose concatenation is γ) and the supplementary angles of the angles between consecutive subarcs at their common endpoints, allows the total curvature of a curve in a space of curvature bounded above to be defined. Again, we refer to [12] for details.

Recall that in a metric space, a sequence of curves γ_m converges to the curve γ if and only if the curves γ and each γ_m admit the parametrizations

$\gamma(t)$ and $\gamma_m(t)$, a$\leq$ t $\leq$ b, such that the sequence of functions γ_m converges uniformly to γ in $[a, b]$ (see [5, p.20]).

Finally, we state some important results that will be used later. One of these is the famous Reshetnyak's majorization theorem (Proposition 3) which is a very useful tool in studying problems in CAT(K) spaces. We will definitely need it in the proof of our main result. For a proper restatement, we first note that a *nonexpanding map* is a map between metric spaces that does not increase the distance between any pair of points. A convex domain D in S_K is said to *majorize* a rectifiable closed curve γ in a metric space if there exists a nonexpanding map from D to that space that maps the boundary of D onto the image of γ in an arclength-preserving way. We also note that the statement of Dekster's estimate (Proposition 4) is a short and simplify version, described in [12], of the results in [10].

Proposition 2. [5, p.30] *Let γ be a rectifiable curve in a metric space. Then for given $\varepsilon > 0$ there exists a $\delta > 0$ such that any subarc of γ of diameter less than δ has length less than ε.*

Proposition 3. [13] *Any closed rectifiable curve of length less than $2\pi/\sqrt{K}$ in a CAT(K) space admits a convex domain in S_K that majorizes it.*

Proposition 4. [10] *For each real number K, there exists a positive number $\theta_K < \pi/2$ such that if $0 \leq \theta \leq \theta_K$ then the maximum length among piecewise C^2 curves in a closed disk of radius less than $\pi/(2\sqrt{K})$ in S_K with total curvature at most θ is finite and attained by a curve with total curvature θ.*

Proposition 5. [5, p.30] (Lower semi-continuity of length.) *If a sequence of curves γ_m converges to a curve γ in a metric space, then $\ell(\gamma) \leq \varliminf_{m\to\infty} \ell(\gamma_m)$.*

Proposition 6. [11, 12] *Suppose σ_n is a sequence of polysegments inscribed in γ such that $\mu_\gamma(\sigma_n) \to 0$. Then $\ell(\sigma_n) \to \ell(\gamma)$, $\sigma_n \to \gamma$ pointwise and $\kappa(\sigma_n) \to \kappa(\gamma)$. Furthermore, if $\kappa(\gamma)$ is finite, then γ is rectifiable.*

Notice that by Proposition 2 we obtain immediately that $\tilde{\mu}_\gamma(\sigma_n) \to 0$ if $\mu_\gamma(\sigma_n) \to 0$.

3 Main Results

In this section, we prove a generalization of the lower semi-continuity of total curvature for curves in Euclidean space to curves in spaces of curvature bounded above. Firstly, we need some lemmas. A proof of the first lemma is a direct generalization of the proof given in [2].

Lemma 1. *In a CAT(0) space, if a polysegment σ is inscribed in a polysegment γ, then $\kappa(\sigma) \leq \kappa(\gamma)$.*

Proof It suffices to show that adjoining one vertex to a polysegment does not decrease its total curvature. Suppose $p_0, p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n$ are vertices of σ and adjoin p_k to obtain ordered vertices $p_0, p_1, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_n$ of γ. Denote by α_i and $\tilde{\alpha_i}$ the angles at p_i of γ and σ, respectively. Let β_{k-1} be the angle at p_{k-1} between the outgoing edges of σ and γ and β_{k+1} be the angle at p_{k+1} between the coming edges. If $1 < k < n-1$,then by the triangle inequality for angles and the excess of a triangle $\triangle(p_{k-1}, p_k, p_{k+1})$,

$$\begin{aligned}\kappa(\gamma) - \kappa(\sigma) &= (\tilde{\alpha}_{k-1} - \alpha_{k-1}) + (\pi - \alpha_k) + (\tilde{\alpha}_{k+1} - \alpha_{k+1}) \\ &\geq -\beta_{k-1} + (\pi - \alpha_k) - \beta_{k+1} \ \geq\ 0.\end{aligned}$$

If $k = 1$, similarly, we obtain

$$\begin{aligned}\kappa(\gamma) - \kappa(\sigma) &= (\pi - \alpha_1) + (\tilde{\alpha}_2 - \alpha_2) \\ &\geq (\pi - \alpha_1) - \beta_2 \ \geq\ 0.\end{aligned}$$

If $k = n-1$, then

$$\begin{aligned}\kappa(\gamma) - \kappa(\sigma) &= (\tilde{\alpha}_{n-2} - \alpha_{n-2}) + (\pi - \alpha_{n-1}) \\ &\geq -\beta_{n-2} + (\pi - \alpha_{n-1}) \ \geq\ 0.\end{aligned}$$

In each case, we conclude that $\kappa(\sigma) \leq \kappa(\gamma)$ as required. □

Remark 1. It follows from the previous lemma that for a curve γ in a CAT(K) space with $K \leq 0$, its total curvature is equal to the supremun of $\kappa(\sigma)$ over all polysegments σ inscribed in γ, that is;

$$\kappa(\gamma) = \lim_{\varepsilon \to 0^+} \sup_{\sigma \in \Sigma_\varepsilon(\gamma)} \kappa^*(\sigma) = \sup_{\sigma \in \Sigma(\gamma)} \kappa(\sigma).$$

Lemma 2. *Let σ be a polysegment in a CAT(K) space with order vertices $\sigma(a) = p_0, p_1, \ldots, p_k = \sigma(b)$. Suppose σ_m is a sequence of polysegments in a CAT(K) space with ordered vertices $\sigma_m(a) = p_0^{(m)}, p_1^{(m)}, \ldots, p_k^{(m)} = \sigma_m(b)$ such that $p_i^{(m)} \to p_i$ for $0 \leq i \leq k$. Then $\kappa(\sigma) \leq \liminf_{m\to\infty} \kappa(\sigma_m)$.*

Proof For $1 \leq i \leq k-1$, write $\alpha_i = \angle p_{i-1}p_ip_{i+1}$ *and* $\alpha_i^{(m)} = \angle p_{i-1}^{(m)}p_i^{(m)}p_{i+1}^{(m)}$. It follows that, $\kappa(\sigma) = \sum_{i=1}^{k-1}(\pi - \alpha_i)$ and $\kappa(\sigma_m) = \sum_{i=1}^{k-1}(\pi - \alpha_i^{(m)})$. Since $p_i^{(m)} \to p_i$ for $0 \leq i \leq k$, we have by Proposition 1 that $\limsup_{m\to\infty} \alpha_i^{(m)} \leq \alpha_i$. Thus,

$$\begin{aligned}\kappa(\sigma) &= \sum_{i=1}^{k-1}(\pi - \alpha_i) \quad \leq \quad \sum_{i=1}^{k-1}(\pi - \limsup_{m\to\infty} \alpha_i^{(m)}) \\ &\leq \sum_{i=1}^{k-1}\pi - \limsup_{m\to\infty}(\sum_{i=1}^{k-1}\alpha_i^{(m)}) \quad = \quad \liminf_{m\to\infty}\sum_{i=1}^{k-1}(\pi - \alpha_i^{(m)}) \\ &= \liminf_{m\to\infty} \kappa(\sigma_m).\end{aligned}$$

□

To prove our theorem, we follow [11, 12] with modifications by using the chord-curvature length estimate for a nonrectifiable case and using the technique of inscribing a sequence of polysegments in and arbitrary close to the sequence of curves being considered for a rectifiable case.

Theorem 1. *If a sequence of curves γ_m converges to a curve γ in a CAT(K) space, then*

$$\kappa(\gamma) \ \leq \ \liminf_{m\to\infty} \kappa(\gamma_m).$$

Proof Let $\gamma, \gamma_m : [a,b] \to X$ be curves in a CAT(K) space such that γ_m converges uniformly to γ in [a,b]. There are two cases to be considered.

case 1. γ is nonrectifiable. Suppose first that γ is contained in a *small* closed ball of radius $R < \pi/2\sqrt{K}$. Now, $\kappa(\gamma) = \infty$ by Proposition 6. Because $\gamma_m \to \gamma$, the property of lower semi-continuity of length yields $\ell(\gamma) \ \leq \ \lim_{m\to\infty} \ell(\gamma_m)$ and hence $\ell(\gamma_m) \to \infty$. We shall show that $\kappa(\gamma_m) \to \infty$. Suppose on the contrary that there are $k > 0$ and a subsequence (γ_{m_i}) of (γ_m) such that $\kappa(\gamma_{m_i})$ is uniformly bounded above by k for all i. Nevertheless, $\ell(\gamma_{m_i}) \to \infty$. Since $\gamma_m \to \gamma$, there is R', $R < R' < \pi/2\sqrt{K}$, such that for sufficiently large m, γ_m is contained in the closed ball of radius R'. For each m, choose a sequence τ_{mj} of polysegments in this closed ball such that $\mu_{\gamma_m}(\tau_{mj}) \to 0$, so that $\tau_{mj} \to \gamma_m$, $\kappa(\tau_{mj}) \to \kappa(\gamma_m)$ and $\ell(\tau_{mj}) \to \ell(\gamma_m)$. But then for each m we have that $\kappa(\tau_{mj}) < \kappa(\gamma_m) + 1$ for sufficiently large j. This means that for sufficiently large i, γ_{m_i} is contained in the closed ball of radius R' and $\kappa(\tau_{m_i j}) < \kappa(\gamma_{m_i}) + 1 \leq k + 1$ for sufficiently large j. Letting j_i be such a large integer j for each i, it is possible to construct a sequence $\sigma_i = \tau_{m_i j_i}$ such that $\ell(\sigma_i) \to \ell(\gamma) = \infty$ and $\kappa(\sigma_i) \leq k + 1$ for all i. By a line of arguments similar to that used in case I of the proof of Proposition 2.4 in [12], this leads to a contradiction. Note that this is where we make use of Proposition 4. Therefore, $\kappa(\gamma_m) \to \infty$ and hence $\kappa(\gamma) \ \leq \ \liminf_{m\to\infty} \kappa(\gamma_m)$.

Now suppose γ is not contained in a small circumball as above. Then we take any R satisfying such a radius condition. The open balls of radii R centered at points on γ form an open cover of the image of γ. By compactness of the image of γ, we can subdivide γ into finitely many subarcs each of which is contained in a circumball satisfying the radius condition. Each γ_m is also subdivided accordingly so that each sequence of subarcs converges to the corresponding subarc of γ. Applying the arguments as above to a nonrectifiable subarc of γ, we obtain a similar contradiction.

case 2. γ is rectifiable. Let σ be a polysegment inscribed in γ and $\sigma(a) = p_0, p_1, \ldots, p_k = \sigma(b)$ be its ordered vertices on γ. Because $\gamma_m \to \gamma$, it is possible to find, for each m, a finite sequence of points $\gamma_m(a) = p_0^{(m)}, p_1^{(m)}, \ldots, p_k^{(m)} = \gamma_m(b)$ on γ_m such that $p_i^{(m)} \to p_i$ as $m \to \infty$ for $0 \leq i \leq k$. For each m, let σ'_m be a polysegment inscribed in γ_m with the ordered vertices of the foregoing points $p_0^{(m)}, p_1^{(m)}, \dots, p_k^{(m)}$. Then $\sigma'_m \to \sigma$ and $\kappa(\sigma) \le \liminf_{m\to\infty} \kappa(\sigma'_m)$ by lemma 2.

Suppose $K \le 0$, then by Remark 1, $\kappa(\gamma_m) = \sup_{\eta\in\Sigma(\gamma_m)} \kappa(\eta)$ and hence $\kappa(\sigma'_m) \le \kappa(\gamma_m)$. According to $\kappa(\sigma) \le \liminf_{m\to\infty} \kappa(\sigma'_m)$, we get that $\kappa(\sigma) \le \liminf_{m\to\infty} \kappa(\gamma_m)$. Because σ is an arbitrary polysegment inscribed in γ we have,

$$\kappa(\gamma) = \sup_{\sigma\in\Sigma(\gamma)} \kappa(\sigma) \le \liminf_{m\to\infty} \kappa(\gamma_m)$$

as desired.

Now suppose $K > 0$. Let σ_n be a sequence of polysegments inscribed in γ such that $\mu_\gamma(\sigma_n) \to 0$ and $\tau_\ell^{(m)}$ a sequence of polysegments inscribed in γ_m such that $\mu_{\gamma_m}(\tau_\ell^{(m)}) \to 0$. Then, by Proposition 6 we get $\sigma_n \to \gamma$, $\tau_\ell^{(m)} \to \gamma_m$, $\kappa(\sigma_n) \to \kappa(\gamma)$, and $\kappa(\tau_\ell^{(m)}) \to \kappa(\gamma_m)$. We also obtain that $\tilde{\mu}_\gamma(\sigma_n) \to 0$, $\tilde{\mu}_{\gamma_m}(\tau_\ell^{(m)}) \to 0$.

Fix n, put $\sigma_n = \sigma$ and $\sigma'_m(n) = \sigma'_m$ as above. Then for each m, a polysegment σ'_m with ordered vertices $\gamma_m(a) = p_0^{(m)}, p_1^{(m)}, \dots, p_k^{(m)} = \gamma_m(b)$ is inscribed in γ_m, $\sigma'_m \to \sigma$ and $\kappa(\sigma) \le \liminf_{m\to\infty} \kappa(\sigma'_m)$. Because $\tau_\ell^{(m)} \to \gamma_m$, for each ℓ there is a finite sequence of points $\tau_\ell^{(m)}(a) = p_0^{(m)} = p_{0\ell}^{(m)}, p_{1\ell}^{(m)}, \dots, p_{k\ell}^{(m)} = p_k^{(m)} = \tau_\ell^{(m)}(b)$ on $\tau_\ell^{(m)}$ such that $p_{i\ell}^{(m)} \to p_i^{(m)}$ for $1 \le i \le k-1$.

For each ℓ, let $\tau_\ell'^{(m)} = \tau_\ell'^{(m)}(n)$ be a polysegment inscribed in $\tau_\ell^{(m)}$ with ordered vertices $p_0^{(m)} = p_{0\ell}^{(m)}, p_{1\ell}^{(m)}, \dots, p_{k\ell}^{(m)} = p_k^{(m)}$. Then, since the vertices of $\tau_\ell'^{(m)}$ converges to the vertices of σ'_m, we have $\tau_\ell'^{(m)} \to \sigma'_m$ and $\kappa(\sigma'_m) \le \liminf_{\ell\to\infty} \kappa(\tau_\ell'^{(m)})$ by lemma 2.

Fix m and ℓ. Notice that $\tau_\ell'^{(m)}$ cuts $\tau_\ell^{(m)}$ into k polysegments $\tau_i''^{(m)} = \tau_i''^{(m)}(\ell)$, $1 \le i \le k$, where endpoints are $p_{i-1\,\ell}^{(m)}$ and $p_{i\ell}^{(m)}$. Following the same arguments as in [12] we have that

$$\kappa(\tau_\ell'^{(m)}) \le KA(m,n,\ell) + \kappa(\tau_\ell^{(m)}),$$

where $A(m,n,\ell) = \displaystyle\sum_{i=1}^{k} a_i(m,n,\ell)$ and each $a_i(m,n,\ell)$ is the area of the convex region in S_K bounded by the convex polygon $P_i(m,n,\ell)$ in S_K that majorizes the closed curve formed by $\tau_{i\ell}''^{(m)}$ and its chord. Since $\lim_{m,n\to\infty} \tilde{\mu}_{\gamma_m}(\sigma'_m(n)) = 0$ and $\lim_{\ell\to\infty} \tau_\ell'^{(m)}(n) = \sigma'_m(n)$ such a polysegment exists by Reshetnyak's majorization theorem if m, n and ℓ are sufficiently large. Notice that, for any partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$ and $s, t \in [t_{i-1}, t_i], 1 \le i \le k$,

$$d(\gamma_m(s), \gamma_m(t)) \le d(\gamma_m(s), \gamma(s)) + d(\gamma(s), \gamma(t)) + d(\gamma(t), \gamma_m(t)).$$

Because $\gamma_m \to \gamma$ and $\mu_\gamma(\sigma_n) \to 0$ we have $\lim\limits_{m,n\to\infty} \mu_{\gamma_m}(\sigma'_m(n)) = 0$, and hence $\lim\limits_{m,n\to\infty} \tilde{\mu}_{\gamma_m}(\sigma'_m(n))$=0.

Write $A(m,n) = \limsup\limits_{\ell\to\infty} A(m,n,\ell)$. With the same reason as in [12] and the fact that $\lim\limits_{m,n\to\infty} \tilde{\mu}_{\gamma_m}(\sigma'_m(n)) = 0$, we have that $\lim\limits_{m,n\to\infty} A(m,n) = 0$. Consequently,

$$\begin{aligned}
\kappa(\sigma_n) &\leq \liminf_{m\to\infty} \kappa(\sigma'_m(n)) \\
&\leq \liminf_{m\to\infty} \liminf_{\ell\to\infty} \kappa(\tau'^{(m)}_\ell) \\
&\leq \liminf_{m\to\infty} \limsup_{\ell\to\infty} KA(m,n,\ell) + \liminf_{m\to\infty} \liminf_{\ell\to\infty} \kappa(\tau^{(m)}_\ell) \\
&= \liminf_{m\to\infty} KA(m,n) + \liminf_{m\to\infty} \kappa(\gamma_m).
\end{aligned}$$

Because $\lim\limits_{m,n\to\infty} A(m,n) = 0$, it follows that $\lim\limits_{n\to\infty} \liminf\limits_{m\to\infty} A(m,n) = 0$. By letting $n \to \infty$,

$$\kappa(\gamma) = \lim_{n\to\infty} \kappa(\sigma_n) \leq \liminf_{m\to\infty} \kappa(\gamma_m),$$

as required. □

Corollary 1. *If a sequence of curves γ_m converges to a curve γ in a space of curvature bounded above, then*

$$\kappa(\gamma) \;\leq\; \liminf_{m\to\infty} \kappa(\gamma_m).$$

Proof Take a finite subcover of the open cover of (the image of) γ consisting of CAT(K) neighborhoods of its points, and apply the theorem. □

Corollary 2. *If a curve γ in a space of curvature bounded above has finite total curvature, then for any point p on γ the total curvature of subarc of γ starting at p converges to 0 as the other endpoint tends to p along γ.*

Proof We only need to consider subarcs of γ in a CAT(K) neighborhood of p. See a proof in Euclidean case in [5, p.121]. The arguments for arbitrary curves described there apply to our case as well. □

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