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OUTCOME-SPACE OUTER APPROXIMATION ALGORITHM FOR LINEAR MULTIPLICATIVE PROGRAMMING

Nguyen T. Bach Kim[*†]
Nguyen T. Le Trang[*] and **Tang T. Ha Yen**[‡]

[*] *Faculty of Applied Mathematics and Informatics*
Hanoi University of Technology, Vietnam
e-mail: kimntb-fami@mail.hut.edu.vn

[‡] *Hanoi Institute of Mathematics, Hanoi, Vietnam*

Abstract

This paper presents an outcome-space outer approximation algorithm for globally solving the linear multiplicative programming problem. We prove that the proposed algorithm is finite. To illustrate the new algorithm, we apply it to solve some sample problems.

1 Introduction

Consider the linear multiplicative programming problem

$$\min\{\prod_{j=1}^{p}\langle c^j, x\rangle \ : \ x \in M\}. \qquad (LMP)$$

[†] Corresponding author
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Assume throughout this paper that M is a nonempty polyhedral convex set defined by

$$M = \{x \in \mathbb{R}^n \ : \ \langle a^i, x\rangle \geq b_i \ , \ i = 1, \cdots, m; x \geq 0\} \tag{1}$$

or in matrix form:

$$M = \{x \in \mathbb{R}^n \ : \ Ax \geq b, \ x \geq 0\}, \tag{2}$$

where A is the $m \times n$ matrix of rows a^i and $b \in \mathbb{R}^m$, $p \geq 2$ is an integer, and for each $j = 1, ..., p$, vector $c^j \in \mathbb{R}^n$ satisfies

$$\langle c^j, x\rangle > 0 \ \text{ for all } x \in M. \tag{3}$$

It is well known that Problem (LMP) is a global optimization problem, i.e., Problem (LMP) generally possesses multiple local optimal solutions that are not globally optimal [4]. Furthermore, Problem (LMP) is known to be NP-hard, even when $p = 2$ [13].

Problem (LMP) has a variety of important applications in engineering, finance, bond portfolio optimization, VLSI chip design and other fields. In recent years, a growing interest in Problems (LMP) has been evident among both researchers and practitioners. Many algorithms have been proposed for globally solving this problem; see, e.g, [1],[4],[7],[11],[15],... and references therein. For a survey of these and related results see [4].

Let C denote the $p \times n$ matrix whose j^{th} row equals c^j, $j = 1, 2, \cdots, p$. The *outcome set* N for problem (LMP) is

$$N = \{y \in \mathbb{R}^p \ : \ y = Cx, \text{ for some } x \in M\}.$$

From [14], N is also a nonempty, polyhedral convex set. One of the most common outcome space reformulations of problem (LMP) is given by the problem

$$\min\{\prod_{j=1}^{p} y_j \ : \ y \in N\}. \tag{$OLMP$}$$

It is easily seen that optimal values of Problems (LMP) and $(OLMP)$ are the same. In this paper, we present an outcome-space outer approximation algorithm for globally solving the linear multiplicative programming problem (LMP). Because p is almost smaller than n, we expect potentially that considerable computational savings could be obtained.

2 Theoretical Prerequisites

First, the existence of global optimal solution of Problem $(OLMP)$ is showed by the next proposition. This fact can be obtained from Proposition 5.1 of [9], however we give here a full proof for the reader's convenience.

Proposition 2.1. *The problem* $(OLMP)$ *always has global optimal solution.*

Proof. As usual, $\mathbb{R}^p_+$ denotes the nonnegative orthant of $\mathbb{R}^p$ and $\text{int}\mathbb{R}^p_+$ is its interior. It is easily seen that the objective function $g(y) = \prod_{j=1}^p y_j$ of problem $(OLMP)$ is increasing on $\text{int}\mathbb{R}^p_+$, i.e., if

$$y^1 \geq y^2 \gg 0 \text{ implies that } g(y^1) \geq g(y^2). \tag{4}$$

Denote the set extreme points of N by N_{ex} and the set of extreme directions of N by N_{ed}. It is well known [14] that

$$N = \text{conv}N_{ex} + \text{cone}N_{ed}, \tag{5}$$

where $\text{conv}N_{ex}$ is the convex hull of N_{ex} and $\text{cone}N_{ed}$ is the cone generated by N_{ed}. Since $\text{conv}N_{ex}$ is a compact set and the function $g(y)$ is continuous on N, there is $y^0 \in \text{conv}N_{ex}$ such that

$$g(\hat{y}) \geq g(y^0), \text{ for all } \hat{y} \in \text{conv}N_{ex}. \tag{6}$$

It is obviously that $y^0 \in N$. We claim that y^0 must be a global optimal solution for problem $(OLMP)$. Indeed, notice under the assumption (3) that

$$\text{cone}N_{ed} \subset (\text{int}R^p_+ \cup \{0\}). \tag{7}$$

For any $y \in N$, it follows from (5) and (7) that

$$y = \bar{y} + v \geq \bar{y}, \tag{8}$$

where $\bar{y} \in \text{conv}N_{ex}$ and $v \in \text{cone}N_{ed}$. Combining (4), (6) and (8) gives

$$g(y) \geq g(\bar{y}) \geq g(y^0).$$

In other words, y^0 is a global optimal solution of problem $(OLPM)$. The proof is completed. □

Let v_m and v_n denote the optimal values of problem (LMP) and $(OLMP)$, respectively. The following proposition tells us the relationship between two problems (LMP) and $(OLMP)$.

Proposition 2.2. *If* y^* *is a global optimal solution to problem* $(OLMP)$*, then any* $x^* \in M$ *such that* $Cx^* = y^*$ *is a global optimal solution to problem* (LMP)*. Furthermore,* $v_n = v_m$*.*

Proof. This follows directly from the definition. □

By Proposition 2.2, instead of solving problem (LMP) we solve problem $(OLMP)$. In many applications, p is much smaller than n. It leads that N

has both smaller dimension and simpler structure than M, so computational savings could be obtained.

For a given nonempty set $Q \subset \mathbb{R}^p$, a point $q^0 \in Q$ is an *efficient point* (or *Pareto point*) of Q if there is no $q \in Q$ satisfying $q^0 > q$, i.e. $Q \cap (q^0 - \mathbb{R}^p_+) = \{q^0\}$. Similarly, a point $q^0 \in Q$ is a *weakly efficient point* if there is no $q \in Q$ satisfying $q^0 \gg q$, i.e. $Q \cap (q^0 - \mathrm{int}\mathbb{R}^p_+) = \emptyset$. We denote Min$Q$ and WMin$Q$ the set of all efficient points of Q and the set of all weakly efficient points of Q, respectively. By the definition,

$$\mathrm{Min}Q \subseteq \mathrm{WMin}Q.$$

Let us recall that the orders in $\mathbb{R}^p$ are defined as follows: $y^1 = (y^1_1, ..., y^1_p)$, $y^2 = (y^2_1, ..., y^2_p) \in \mathbb{R}^p$,

$$\begin{aligned}
&y^1 \geq y^2 \text{ if } y^1_i \geq y^2_i \text{ for all } i = 1, ..., p;\\
&y^1 > y^2 \text{ if } y^1 \geq y^2 \text{ and } y^1 \neq y^2;\\
&y^1 \gg y^2 \text{ if } y^1_i > y^2_i \text{ for all } i = 1, ..., p.
\end{aligned}$$

The following result (Theorem 2.5, Chapter 4 [12]) will be used (see also Theorem 2.1.5 [17])

Proposition 2.3. *Let the set $Q \subset \mathbb{R}^p$. A point $y^0 \in Q$ is a weakly efficient of Q if and only if there is a nonzero vector $p \in \mathbb{R}^p$ and $p \geq 0$ such that y^0 is an optimal solution to the linear programming problem*

$$\min\{\langle p, y\rangle \ : \ y \in Q\}.$$

Remark 2.1. Invoking the assumption (3), it is easily seen that Min$N$ is nonempty.

It is well known that the objective function $g(y) = \prod_{j=1}^p y_j$ of problem $(OLMP)$ is a quasiconcave on N and attains its minimum at an extreme point of N (see [4]). Combining this fact and the definition of an efficient point gives the following result which will be needed.

Proposition 2.4. *Any global optimal solution to problem* $(OLMP)$ *must belong to the efficient extreme point set* Min$N \cap N_{ex}$.

By Proposition 2.4, one can find global optimal solutions for problem $(OLMP)$ by determining the set of all efficient extreme points of N and comparing the values of the objective function at these efficient extreme points. Some algorithms for generating Min$N \cap N_{ex}$ have been proposed, see, for example,[2],[3],[5],[10].

Here, it is worth noticing that the new algorithm allows us to find a global optimal solution to problem $(OLMP)$ without determining the whole set Min$N \cap N_{ex}$ (see Remark 3.1 in Section 3).

Denote by $y^{lo} = (y_1^{lo}, ..., y_p^{lo})$, where for each $j = 1, 2, ..., p$, y_j^{lo} equals to the minimum value of the linear programming

$$\min\{y_j \ : \ y \in N\}. \qquad (L_j^{lo})$$

Notice that y^{lo} generally do not belong to N. If $y^{lo} \in N$ then Min$N = \{y^{lo}\}$ and y^{lo} is the global solution to problem $(OLMP)$. We therefore assume henceforth that $y^{lo} \notin N$.

Denote the optimal solution of the problem (L_j^{lo}) by $v^j = (v_1^j, ..., v_p^j)$, $j = 1, ..., p$. Let

$$v_M = \max\{v_i^j, \ j = 1, ..., p;, \ i = 1, ..., p\}$$

and

$$y^{up} = (y_1^{up}, ..., y_p^{up}), \text{ with } y_j^{up} = \alpha > v_M \text{ for all } j = 1, ..., p.$$

Consider the set N^{co} defined by

$$N^{co} = (N + \mathbb{R}^p_+) \cap (y^{up} - \mathbb{R}^p_+).$$

It is clear that N^{co} is a nonempty, full-dimension compact polyhedron in $\mathbb{R}^p_+$.

Proposition 2.5. Min$N$ = MinN^{co}.

Proof. ($\Rightarrow$) We will begin with showing that Min$N \subseteq$ MinN^{co}. Let $y^* \in$ Min$N$. By definitions, we have $y^* \in N \subset N + \mathbb{R}^p_+$ and $y^* < y^{up}$. This implies that $y^* \in N^{co}$. If $y^* \notin$ MinN^{co} then there exists $\bar{y} \in N^{co}$ such that $y^* > \bar{y}$. Since $N^{co} \subset N + \mathbb{R}^p_+$, we have $\bar{y} = y^0 + u$ where $y^0 \in N$ and $u \geq 0$. Therefore, $y^* > y^0$ which contradicts the fact $y^* \in$ Min$N$. It implies that $y^* \in$ MinN^{co}.

($\Leftarrow$) We now prove that Min$N \supseteq$ MinN^{co}. Let $y^* \in$ MinN^{co}. First, we show that $y^* \in N$. Indeed, since $y^* \in N^{co}$, by definition of N^{co} we have $y^* = y^0 + u = y^{up} - v$ where $y^0 \in N$, $\ u \geq 0$ and $v \geq 0$. If $u > 0$ then $y^0 = y^{up} - (v + u) \in (y^{up} - \mathbb{R}^P_+)$. Hence, $y^0 \in N^{co}$ and $y^* > y^0$. Since $y^* \in$ MinN^{co}, we have $y^* = y^0 \in N$. To complete the proof it remains to show that $y^* \in$ Min$N$. Assume the contrary that $y^* \notin$ Min$N$. By definitions, there is $\bar{y} \neq y^*$, $\bar{y} \in N$ such that $\bar{y} > y^*$, i.e., $\bar{y} = y^* - v$ with $v > 0$. As $y^* \in N^{co}$, we have $y^* = y^{up} - t$ and $t \leq 0$. Thus $\bar{y} = y^* - v = y^{up} - t - v = y^{up} - (t + v)$, where $(t + v) > 0$. That means $\bar{y} \in N^{co}$ and $y^* > \bar{y}$. This contradict to that $y^* \in$ MinN^{co}. This proof is completed. $\square$

Let

$$\begin{aligned} B^0 &= (y^{lo} + \mathbb{R}^p_+) \cap (y^{up} - \mathbb{R}^p_+) \\ &= \{y \in \mathbb{R}^p \ : \ y^{lo} \leq y \leq y^{up}\}. \end{aligned}$$

We have $N^{co} \subset B^0$ and Min$B^0 = \{y^{lo}\}$. It is clear that the set of all extreme points of B^0 can be easily determined.

Starting with the box B^0, the outer approximation algorithm will iteratively generate a finite number of nonempty, compact, polyhedra B^k, $k = 0, 1, 2, \cdots$ such that

$$B^0 \supset B^1 \supset B^2 \supset \cdots \supset N^{co}.$$

In a typical iteration k, the polyhedra B^{k+1} is defined by

$$B^{k+1} = B^k \cap \{y \in \mathbb{R}^p \ : \ \langle p^*, y\rangle \geq \langle p^*, y^k\rangle\},$$

where y^k is the intersection between the line segment $[v^k, y^{up}]$ and the boundary of N^{co}, v^k is a vector belonging to the set $B^k \setminus N^{co}$ and $p^* \in \mathbb{R}^p$ is a nonzero nonnegative vector. Furthermore,

i) The following Proposition 2.6 shows that the point y^k is a weakly efficient point of the set N^{co} (i.e, $y^k \in \mathrm{WMin}N^{co}$).

ii) The separation hyperplane

$$\{y \in R^p \ : \ \langle p^*, y\rangle = \langle p^*, y^k\rangle\},$$

which is analogous to Benson's the separation hyperplane (see Theorem 2.5 [3]), can be determined by the Proposition 2.7.

Remark 2.2. In a typical iteration k, we have $N^{co} \subset B^k$. The definition leads to the relation

$$\mathrm{WMin}B^k \cap N^{co} \subset \mathrm{WMin}N^{co}.$$

Remark 2.3. Since the vector p^* is nonzero nonnegative, by Proposition 2.3, the set

$$B^{k+1} \cap \{y \in R^p \ : \ \langle p^*, y\rangle = \langle p^*, y^k\rangle\} \subset \mathrm{WMin}B^{k+1}.$$

Therefore

$$VB^{k+1}_{new} := B^{k+1}_{ex} \setminus B^k_{ex} \subset \mathrm{WMin}B^{k+1},$$

where B^k_{ex} denotes the set of all extreme points of B^k.

Proposition 2.6. *For any $\bar{v} \in B^k \setminus N^{co}$, the line segment $[\bar{v}, y^{up}]$ contains a unique point $y^w \in \mathrm{WMin}N^{co}$.*

Proof. By the convexity of the line segment $[\bar{v}, y^{up}]$ and the set N^{co} we have the unique point y^w belongs to $[\bar{v}, y^{up}] \cap \partial N^{co}$. Now we show that $y^w \in \mathrm{WMin}N$. Since N^{co} is a compact polyhedron and y^w belongs to the boundary of N^{co}, the set $A = N^{co} - y^w$ is also a compact polyhedron containing the origin 0 of the space $\mathbb{R}^p$ and 0 belongs to the boundary of A. It is well known (see Separation Theorems [14]) that there is a nonzero vector p such that

$$\langle p, u\rangle \geq 0 \text{ for all } u \in A. \tag{9}$$

Then, it is easy to show that

$$\langle p, v\rangle \geq 0 \text{ for all } v \in \text{cone}A, \tag{10}$$

where

$$\text{cone}A = \{v = tu \ : \ u \in A, t \geq 0\} \tag{11}$$

is the cone generated by A. From (10) and (11), since $\bar{u} = y^{up} - y^w \in A$, we have

$$\langle p, t\bar{u}\rangle = t\langle p, \bar{u}\rangle \geq 0 \text{ for all } t \geq 0. \tag{12}$$

Notice that by definition, we have $\bar{u} \gg 0$. Therefore (12) is only true when

$$p \geq 0. \tag{13}$$

From the definition of A and (9) we deduce

$$\langle p, y - y^w\rangle \geq 0 \text{ for all } y \in N^{co},$$

i.e.,

$$\langle p, y\rangle \geq \langle p, y^w\rangle \text{ for all } y \in N^{co}. \tag{14}$$

Combining Proposition 2.3, (13) and (14), the proof is straight-forward. □

Proposition 2.7. *Assume that $y^w \in \text{WMin}N^{co}$. Denote by (p^{*T}, u^{*T}) an optimal solution to the following linear programming problem*

$$\begin{array}{rl} \max & -\langle y^w, p\rangle + \langle b, u\rangle, \\ \text{subject to} & -p^TC + u^TA \leq 0, \\ & \langle e, p\rangle \qquad\quad \geq 1, \\ & \qquad\qquad p, u \geq 0, \end{array} \tag{DT}$$

where $e \in \mathbb{R}^p$ is the vector in which each entry equal to 1.0, $p \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$. Then $p^* \geq 0$, $p^* \neq 0$ and y^w belongs to a weakly efficient face of N^{co} given by

$$\{y \in N^{co} \ : \ \langle p^*, y\rangle = \langle b, u^*\rangle\}.$$

Proof. Consider the following linear programming problem

$$\begin{array}{rl} \max & t \\ \text{subject to} & Cx + et \leq y^w, \\ & Ax \qquad \geq b, \\ & x \geq 0, \ t \geq 0. \end{array}$$

Since $y^w \in \text{WMin}N^{co}$, it can easily be seen that the optimal value of this problem equals to zero. For convenience, we restate this problem in an equivalent form

$$\min \ -t \tag{T}$$

$$\begin{aligned} \text{subject to} \quad & Cx + et \le y^w, \\ & Ax \qquad \ge b, \\ & x \ge 0,\ t \ge 0. \end{aligned}$$

Let vo_T be the optimal value of problem (T). It is clear that $vo_T = 0$. The dual linear programming problem of problem (T) is given by

$$\begin{aligned} \max\ & \langle y^w, \bar{p}\rangle + \langle b, u\rangle, \\ \text{subject to}\ & \bar{p}^T C + u^T A \le 0, \\ & \langle e, \bar{p}\rangle \qquad \le -1, \\ & \bar{p} \le 0,\ u \ge 0. \end{aligned} \tag{DT_0}$$

Let $p = -\bar{p}$. It is easy to see that problem (DT_0) becomes problem (DT). That means problem (DT) and problem (T) are dual each other. Denote by vo_{DT} the optimal value of problem (DT). By the dual theory of linear programming, $vo_T = vo_{DT}$. Hence, we have $vo_{DT} = 0$, i.e.

$$\langle y^w, p^*\rangle = \langle b, u^*\rangle, \tag{15}$$

where $({p^*}^T, {u^*}^T)$ is an optimal solution to problem (DT). Because $({p^*}^T, {u^*}^T)$ is a feasible solution to problem (DT), we have $p^* \ge 0$ and $p^* \ne 0$.

Therefore, in view of Proposition 2.3, the optimal solution set of the linear programming

$$\min\langle\{p^*, y\rangle \ :\ y \in N^{co}\} \tag{W}$$

is an weakly efficient face of N^{co}. To complete the proof it remains to show that y^w belongs to this weakly efficient face.

The explicit form to problem (W) is

$$\begin{aligned} \min\ & \langle p^*, y\rangle \\ \text{subject to}\ & y \qquad \le y^{up}, \\ & -y + Cx \le 0, \\ & Ax \ge b, \\ & x \ge 0. \end{aligned} \tag{P_W}$$

The dual linear programming problem of (P_W) is given

$$\begin{aligned} \max\ & \langle y^{up}, s\rangle + \langle b, q\rangle, \\ \text{subject to}\ & s^T - r^T \qquad = {p^*}^T, \\ & r^T C + q^T A \le 0, \\ & r,\ s \le 0,\ q \ge 0. \end{aligned} \tag{DP_W}$$

Checking directly shows that

(i) $(s^T, r^T, q^T) = (0^T, -{p^*}^T, u^*)$ is a feasible solution to problem (DP_W) and the respective objective function value is $\langle b, u^*\rangle$;

(ii) $(y^T, x^T) = ({y^w}^T, {x^*}^T)$ is a feasible solution to problem (P_W) and the respective objective function value is $\langle p^*, y^w\rangle$;

Furthermore, from (15) we have

(iii) $\langle b, u^* \rangle = \langle p^*, y^w \rangle$.

Combining (i), (ii) and (iii) by duality theory of linear programming leads the fact that y^w is an optimal solution to problem (P_W). This concludes the proof. $\square$

3 The Algorithm

By virtue of Proposition 2.2, the solution of Problem (LMP) will be carried out in two stages:

i) Determining a global optimal solution to Problem $(OLMP)$;

ii) For each global optimal solution $y^* \in N$ to problem $(OLMP)$, finding a global optimal solution $x^* \in M$ to problem (LMP) that satisfies $Cx^* = y^*$. To accomplish this, we can solve the following linear system

$$\begin{cases} Cx = y^* \\ Ax \geq b \\ x \geq 0. \end{cases}$$

3.1 Outcome-Space Outer Approximation Algorithm

The algorithm for solving Problem (LMP) can be described as follows

Phase 1. (*Finding a global optimal solution to problem* $(OLMP)$)

Initialization step. Determine the points y^{lo} and y^{up}. Start with the box

$$B^0 = \{y \in \mathbb{R}^p \ : \ y_i^{lo} \leq y_i \leq y_i^{up}, i = 1, ..., p\}.$$

The vertex set $V(B^0)$ of B^0 can easily be determined.

Set $VB^0_{new} = \{y^{lo}\}$ (*we have* $\text{Min}B^0 = VB^0_{new}$) and $k = 0$.

Iteration k, $k = 0, 1, 2, ...$ See Steps $k1$ through $k5$ below

Step k1. Determine the optimal solution set

$$B^{opt} = \text{argmin}\{g(v), v \in VB^k_{new}\}.$$

Set $\bar{B} = B^{opt} \cap \text{Min}N$.

If $\bar{B} \neq \emptyset$ (*every* $y^* \in \bar{B}$ *is a solution to Problem* $(OLMP)$) **Then** Goto Phase 2 **Else** Go to Step $k2$.

Step k2. Choose an arbitrary $v^k \in B^{opt} \setminus N^{co}$. Determine

$$y^k \in [v^k, y^{up}] \cap \partial N^{co},$$

where ∂N^{co} denotes the boundary of N^{co}.

Step k3. Find an optimal solution $({p^*}^T, {u^*}^T)$ to the linear programming problem (DT) with $y^w = y^k$.

Step k4. Set

$$B^{k+1} = \{y \in B^k \ : \ \langle p^*, y\rangle \geq \langle b, u^*\rangle\},$$

and determine the set $VB^{k+1}_{new} = B^{k+1}_{ex} \backslash B^k_{ex}$. (*By Remark 2.3, we have* $VB^{k+1}_{new} \subset$ WMinB^{k+1})

Step k5. Set $k := k+1$ and go to Iteration k.

Phase 2. (*Finding a global optimal solution to problem* (LMP))
For each $y^* \in \bar{B}$, find a point $x^* \in M$ such that $Cx^* = y^*$. Then x^* is a global solution to problem (LMP).

Below, we will show the finiteness of the above algorithm.

Proposition 3.1. *The outcome-space outer approximation algorithm is finite.*

Proof. The algorithm start from the box B^0. In every Step $k4$ of the iteration k, we have the box

$$B^{k+1} = \{y \in B^k \ : \ \langle p^*, y\rangle \geq \langle b, u^*\rangle\},$$

where, by Proposition 2.7, $\{y \in \mathbb{R}^n \ : \ \langle p^*, y\rangle = \langle b, u^*\rangle\}$ is a weakly efficient face of N^{co}. The algorithm systematically generates distinct polyhedra B^k, k=0,1,2,... such that

$$B^0 \supset B^1 \supset \cdots \supset N^{co}.$$

Since N^{co} is a nonempty compact polyhedra, the algorithm must be finite. $\Box$.

Let us conclude this section with some remarks on the implementation of the computational modules in the above algorithm.

Remark 3.1. (*about checking whether* $y^* \in \mathrm{Min}N$ *in Step* $k1$)
The following multiobjective linear programming problem associated the multiplicative linear programming (LMP)

$$\mathrm{MIN}\{Cx \ : \ x \in M\}. \qquad (MOP)$$

A point $x^0 \in M$ is an *efficient solution* of (MOP) if $y^0 = Cx^0$ is an efficient point of the set N. The following fact can be easily deduced from the definitions.

Proposition 3.2. *Let $y^* \in N$. If $x^* \in X$ satisfies $Cx^* = y^*$ and x^* is an efficient solution to problem (MOP) then y^* is an efficient point of N.*

Now, we rewrite (1) as follows

$$M = \{x \in \mathbb{R}^n \ : \ \langle \bar{a}^i, x \rangle \geq \bar{b}_i, \ \ i = 1, \cdots, m+n\},$$

where $\bar{a}^i = a^i$, $\bar{b}_i = b_i$ for all $i = 1, \cdots, m$ and $\bar{a}^i = e^i$, $\bar{b}_i = 0$ for all $i = m+1, \cdots, m+n$ with e^i is unit vector i^{th}. Then, let us recall from [8] the condition for a point to be an efficient solution to problem (MOP).

Proposition 3.3. (see Corollary 5.4 [8]) *A point $x^* \in M$ is an efficient solution for problem (MOP) if and only if the following system is consistent (has a solution)*

$$\begin{cases} \sum_{j=1}^{p} \lambda_j c^j + \sum_{i \in I(x^*)} \mu_i \bar{a}^i = 0, \\ \lambda_j > 0, \forall i = 1, \cdots, p, \\ \mu_i \geq 0, \forall i \in I(x^*), \end{cases} \tag{16}$$

where

$$I(x^*) = \{i \in \{1, \cdots, n+m\} \ : \ \langle \bar{a}^i, x^* \rangle = \bar{b}_i\}.$$

Proposition 3.2 and Proposition 3.3 allow us to check whether a point $y^* \in \mathbb{R}^p$ is an efficient point of N. It can be executed by the following procedure.

Procedure $EF(y^*)$**;**
Step 1. (*Checking whether $y^* \in N$*)
Solve the following linear system

$$\begin{cases} Cx = y^*, \\ Ax \geq b \\ x \geq 0. \end{cases} \tag{17}$$

If The system (17) has a solution x^* (*i.e.*, $y^* \in N$) **Then** Go to Step 2.
Else Stop. ($y^* \notin N$, *hence* $y^* \notin \text{Min}N$)

Step 2. Solve the system (16)
If The system has a solution **Then** Stop (x^* *is an efficient solution to* (MOP), *hence* $y^* \in \text{Min}N$)
Else Stop. ($y^* \notin \text{Min}N$)

Remark 3.2. (*about finding $y^k \in [v^k, y^{up}] \cap \partial N^{co}$ in Step $k2$*)
To determine $y^k \in [v^k, y^{up}] \cap \partial N^{co}$ we solve the linear programming problem

$$\lambda^* = \min \lambda \tag{P_λ}$$

$$\text{subject to} \begin{cases} Cx + \lambda(v^k - y^{up}) \leq v^k \\ Ax \geq b \\ x \geq 0, \\ 0 < \lambda < 1. \end{cases}$$

Then, we have

$$y^k = (1 - \lambda^*)v^k + \lambda^* y^{up}.$$

Remark 3.3. In Step k4, we have to determine the set $VB_{new}^{k+1} = B_{ex}^{k+1} \setminus B_{ex}^k$. Since B^{k+1} is obtained from B^k by adding a new constraint linear inequality, the set B_{ex}^{k+1} can be calculated from those of B^k by using some existing methods (see, for example,[6],[16]).

3.2 Examples

A test software implementing the algorithm had been constructed in Visual C^{++} programming language. This is a self-contain software. The procedures for solving the subsidiary linear programming problems and for checking whether the system (16) is consistent are based on the well known simplex method.In a typical iteration k, for determining the extreme point set B_{ex}^{k+1} we used an own code based on the algorithm proposed by T.V. Thieu [16].

Example 1 We begin with the following simple example, which illustrates the process of the algorithm. Consider the linear multiplicative programming problem

Example 1.

$$\min\{\langle c^1, x\rangle\langle c^2, x\rangle \mid Ax \geq b,\ x \geq 0\}, \qquad (LMP_{exam})$$

where

$$c^1 = (3 \quad 1\,), \quad c^2 = (0 \quad 1\,), \quad A = \begin{pmatrix} -1 & -3 \\ -2 & 1 \\ 2 & -1 \\ 0 & 1 \\ 1 & 3 \\ 5 & 6 \\ 2 & 1 \end{pmatrix} \quad \text{and,} \quad b = \begin{pmatrix} -30 \\ -18 \\ -3 \\ 1 \\ 9 \\ 30 \\ 8 \end{pmatrix}.$$

The process of computing is as follows.

Phase 1.

Initialization step: Soling linear programming problems (L_1^{lo}) and (L_2^{lo}), we obtain

$$v^1 = (9.25, 5.5),\ v^2 = (19, 1),$$
$$y^{lo} = (9.25, 1) \text{ and } y^{up} = (19.1, 19.1).$$

Set $\quad B^0 = \{y \in \mathbb{R}^2 \ :\ 9.25 \leq y_1 \leq 19.1;\ 1 \leq y_2 \leq 19.1\};$

$$VB^0_{new} = \{y^{lo}\} = (9.25, 1); \quad k := 0.$$

Iteration $k = 0$;

Step 01: Solving $\min\{g(v)\ :\ v \in VB^0_{new}\}$, we obtain $B^{opt} = \{(9.25, 1)\}$ and $\bar{B} = \emptyset$. Then go to Step 02.

Step 02: Choose $v^0 = (9.25, 1)$. Solving problem (P_λ) with $k = 0$ we obtain the optimal value $\lambda^* = 0.1190$ and the optimal solution

$$(x^*, \lambda^*) = (2.4226,\ 3.1547,\ 0.1190).$$

Hence,

$$y^0 = (1 - \lambda^*)v^0 + \lambda^* y^{up} = (10.4226,\ 3.1547).$$

Step 03: Solving linear problem (DT) with $y^w = y^0$, we obtain:

$$p^0 = (0.6667,\ 0.3333),\ u^0 = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1) \ \text{ and } \ \langle b, u^0\rangle = 8.$$

Step 04: Set $B^1 = \{y \in B^0\ :\ 0.6667y_1 + 0.3333y_2 \geq 8\}$.

$$B^1 = \{y \in \mathbb{R}^2\ :\ 9.25 \leq y_1 \leq 19.1;\ 1 \leq y_2 \leq 19.1;\ 0.6667y_1 + 0.3333y_2 \geq 8\}.$$

We have $VB^1_{new} = \{(9.25,\ 5.5);\ (11.5,\ 1)\}$.

Step 05: $k := 1$ and go to iteration 1.

Iteration $k = 1$;

Step 11: Solving $\min\{g(v)\ :\ v \in VB^1_{new}\}$, we obtain $B^{opt} = \{(11.5,\ 1)\}$ and $\bar{B} = \emptyset$. Then go to Step 12.

Step 12: Choose $v^1 = (11.5,\ 1)$. Solving problem (P_λ) with $k = 1$ we obtain the optimal value $\lambda^* = 0.0714$ and the optimal solution

$$(x^*, \lambda^*) = (3.2503,\ 2.2914,\ 0.0714).$$

Hence,

$$y^1 = (1 - \lambda^*)v^1 + \lambda^* y^{up} = (12.0423,\ 2.2914).$$

Step 13: Solving linear problem (DT) with $y^w = y^1$, we obtain:

$$p^1 = (0.2778,\ 0.7222),\ u^1 = (0,\ 0,\ 0,\ 0,\ 0,\ 0.1667,\ 0) \ \text{ and } \ \langle b, u^1\rangle = 5.$$

Step 14: Set
$$B^2 = \{y \in B^1\ :\ 0.2778y_1 + 0.7222y_2 \geq 5\} =$$

$= \{y \in \mathbb{R}^2 \ : \ 9.25 \leq y_1 \leq 19.1;\ 1 \leq y_2 \leq 19.1;$
$0.6667y_1 + 0.3333y_2 \geq 8;\ 0.2778y_1 + 0.7222y_2 \geq 5\}.$

We have $VB^2_{new} = \{(15.4,\ 1),\ (10.5714,\ 2.8571)\}$.

Step 15: $k := 2$ and go to iteration 2.

Iteration $k = 2$;

Step 21: Solving $\min\{g(v) \ : \ v \in VB^2_{new}\}$, we obtain $B^{opt} = \{(15.4,\ 1)\}$ and $\bar{B} = \emptyset$. Then go to Step 22.

Step 22: Choose $v^2 = (15.4,\ 1)$. Solving problem (P_λ) with $k = 2$ we obtain the optimal value $\lambda^* = 0.0242$ and the optimal solution

$$(x^*, \lambda^*) = (4.6836,\ 1.4387,\ 0.0242).$$

Hence,

$$y^2 = (1 - \lambda^*)v^2 + \lambda^* y^{up} = (15.4896,\ 1.4387).$$

Step 23: Solving linear problem (DT) with $y^w = y^2$, we obtain:

$$p^2 = (0.1111,\ 0.8889),\ u^2 = (0,\ 0,\ 0,\ 0,\ 0.3333,\ 0,\ 0) \ \text{ and } \ \langle b, u^2 \rangle = 3.$$

Step 24: Set
$B^3 = \{y \in B^2 \ : \ 0.1111y_1 + 0.8889y_2 \geq 3\} =$
$= \{y \in \mathbb{R}^2 : 9.25 \leq y_1 \leq 19.1; 1 \leq y_2 \leq 19.1; 0.6667y_1 + 0.3333y_2 \geq 8;$
$0.2778y_1 + 0.7222y_2 \geq 5;\ 0.1111y_1 + 0.8889y_2 \geq 3\}.$

We have $VB^3_{new} = \{(19,\ 1),\ (13.667,\ 1.667)\}$.

Step 25: $k := 3$ and go to iteration 3.

Iteration $k = 3$;

Step 31: Solving $\min\{g(v) \ : \ v \in VB^3_{new}\}$, we obtain $B^{opt} = \{(19,\ 1)\}$ and $\bar{B} \neq \emptyset$. In particular, take $y^* = (19,\ 1)$. Use Procedure $EF(y^*)$ with $y^* = (19,\ 1)$ we confirm that $y^* \in \text{Min}N$. Then $\bar{B} = \{(19,\ 1)\}$.
Go to Phase 2.

Phase 2. We have $y^{opt} = (19,\ 1)$ is an optimal solution to problem $(OMLP)$.
Solving the system (17) with $y^* = y^{opt} = (19,\ 1)$, we obtain the optimal solution $x^{opt} = (6,\ 1)$ to problem (LMP).
The algorithm is terminated.

Remark 3.4. In the Example 1, the set of all efficient extreme points of N consists exactly of four points,

$$\text{MinN} \cap N_{ex} = \{(19,\ 1);\ (13.6667,\ 1.6667);\ (10.5714,\ 2.8571);\ (8,\ 2.5)\}.$$

However, in the calculating by the proposed algorithm to obtain the global optimal solution to the problem (LMP_{exam}), in fact, we need to work with three efficient extreme points $(19,\ 1)$, $(13.6671, 1.6671)$ and $(10.5714, 2.8571)$.

Example 2. The following example introduced by H.P. Benson and G.M. Boger [4], and also considered in [7]. The problem is stated as follows.

$$\min\{\langle c^1, x\rangle\langle c^2, x\rangle \mid Ax = b,\ x \geq 0\},$$

where

$c^1 = (1\ \ 0\ \ 1/9\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ),\ \ c^2 = (0\ \ 1\ \ 1/9\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ),$

$$A = \begin{pmatrix} 9 & 9 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 1 & 8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 8 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 7 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 81 \\ 72 \\ 72 \\ 9 \\ 9 \\ 9 \\ 8 \\ 8 \end{pmatrix}.$$

The process of computing is as follows.

Phase 1.

Initialization step: Solving linear programming problems (L_1^{lo}) and (L_2^{lo}) we obtain

$$v^1 = (0.1111, 8.1111),\ \ v^2 = (8.1111, 0.1111)$$
$$y^{lo} = (0.1111, 0.1111), \text{ and } y^{up} = (8.2111, 8.2111).$$

Set $\quad B^0 = \{y \in \mathbb{R}^2\ :\ 0.1111 \leq y_1 \leq 8.2111;\ 0.1111 \leq y_2 \leq 8.2111\};$

$VB^0_{new} = \{y^{lo}\} = (0.1111, 0.1111);\quad k := 0.$

Iteration $k = 0$;

Step 01: Solve $\min\{g(v)\ :\ v \in VB^0_{new}\}$, we obtain $B^{opt} = \{((0.1111, 0.1111)\}$ and $\bar{B} = \emptyset$. Then go to Step 02.

Step 02: Choose $v^0 = (0.1111, 0.1111)$. Solve problem (P_λ) with $k = 0$ we obtain the optimal value $\lambda^* = 0.1097$ and the optimal solution
$(x^*, \lambda^*) = (0,\ 0,\ 9,\ 63,\ 0,\ 0,\ 0,\ 0,\ 54,\ 8,\ 8,\ 0.1097)$. Hence,

$$y^0 = (1 - \lambda^*)v^0 + \lambda^* y^{up} = (0.9999,\ 0.9999).$$

Step 03: Solving linear problem (DT) with $y^w = y^0$, we obtain:

$p^0 = (0.8889,\ 0.1111),\ u^0 = (0,\ 0,\ 0,\ 0.1111,\ 0,\ 0,\ 0,\ 0)$ and $\langle b, u^0\rangle = 0.9999.$

Step 04: Set
$B^1 = \{y \in B^0 \ : \ 0.8889y_1 + 0.1111y_2 \geq 0.9999\}.$
$= \{y \in \mathbb{R}^2 \ : \ 0.1111 \leq y_1 \leq 8.2111;\ 0.1111 \leq y_2 \leq 8.2111;$
$0.8889y_1 + 0.1111y_2 \geq 0.9999\}.$
We have $VB^1_{new} = \{(1.1111,\ 0.1111);\ (0.1111,\ 8.2111)\}$

Step 05: $k := 1$ and go to iteration 1.

Iteration $k = 1$**;**

Step 11: Solving $\min\{g(v)\ :\ v \in VB^1_{new}\}$, we obtain $B^{opt} = \{(1.1111,\ 0.1111)\}$ and $\bar{B} = \emptyset$. Then go to Step 12.

Step 12: Choose $v^1 = (1.1111,\ 0.1111)$. Solving problem (P_λ) with $k = 1$ we obtain the optimal value $\lambda^* = 0.0973$ and the optimal solution

$$(x^*, \lambda^*) = (0.9025, 0, 8.0975, 56.6822, 0, 6.3178, 5.4153, 0, 48.5847, 7.0975, 8, 0.0973).$$

Hence,

$$y^1 = (1 - \lambda^*)v^1 + \lambda^* y^{up} = (1.8022,\ 0.8996).$$

Step 13: Solving linear problem (DT) with $y^w = y^1$, we obtain:

$$p^1 = (0.1111,\ 0.8889),\ u^1 = (0,\ 0,\ 0,\ 0,\ 0.1111,\ 0,\ 0,\ 0)\ \text{ and }\ \langle b, u^1\rangle = 0.9999.$$

Step 14: Set
$B^2 = \{y \in B^1 \ : \ 0.1111y_1 + 0.8889y_2 \geq 0.9999\} =$
$= \{y \in \mathbb{R}^2 \ : \ 0.1111 \leq y_1 \leq 8.2111;\ 0.1111 \leq y_2 \leq 8.2111;$
$0.8889y_1 + 0.1111y_2 \geq 0.9999;\ 0.1111y_1 + 0.8889y_2 \geq 0.9999\}.$
We have $VB^2_{new} = \{(0.9999,\ 0.9999),\ (8.1111,\ 0.1111)\}.$

Step 15: $k := 2$ go to step iteration 2.

Iteration $k = 2$**;**

Step 21: Solving $\min\{g(v)\ :\ v \in VB^2_{new}\}$, we obtain $B^{opt} = \{(8.1111,\ 0.1111)\}$ and $\bar{B} \neq \emptyset$. In particular, Take $y^* = (8.1111,\ 0.1111)$. Using Procedure $EF(y^*)$ with $y^* = (8.1111,\ 0.1111)$ we confirm that $y^* \in \mathrm{Min}N$. Then $\bar{B} = \{(8.1111,\ 0.1111)\}$.
Go to Phase 2.

Phase 2. We have an optimal solution $y^{opt} = (8.1111,\ 0.1111)$ to problem $(OMLP)$.

Solving the system (11) with $y^* = y^{opt} = (8.1111,\ 0.1111)$, we obtain an optimal solution

$$x^{opt} = (8,\ 0,\ 1,\ 7,\ 0,\ 56,\ 48,\ 0,\ 6,\ 0,\ 8)$$

to problem (LMP).

The algorithm is terminated.

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