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A COMPACT EMBEDDING OF SEMISIMPLE SYMMETRIC SPACES

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Abstract

Let G be a connected real semisimple Lie group with finite center and σ be an involutive automorphism of G. Suppose that H is a closed subgroup of G with $G_e^\sigma \subset H \subset G^\sigma$, where G^σ is the fixed points group of σ and G_e^σ denotes its identity component. The coset space $\mathbf{X} = G/H$ is then a semisimple symmetric space. Let θ be a Cartan involution which commutes with σ and K be the set of all fixed points of θ. Then K is a σ-stable maximal compact subgroup of G and the coset space G/K becomes a Riemannian symmetric space of noncompact type. By using the action of the Weyl group, we have constructed a compact real analytic manifold in which the Riemannian symmetric space G/K is realized as an open subset and that G acts analytically on it. The purpose of this note is to apply the above construction to the case of semisimple symmetric spaces $\mathbf{X} = G/H$. Our construction is similar to those of Schlichtkrull, Lizhen Ji, Oshima for Riemannian symmetric spaces and similar to those of Kosters, Sekiguchi, Oshima for semisimple symmetric spaces.

1 Introduction

Let G be a connected real semisimple Lie group with finite center, σ be an involutive automorphism of G and $\mathbf{X} = G/H$ be the corresponding semisimple symmetric space. Here H is a closed subgroup of G with $G_e^\sigma \subset H \subset G^\sigma$, where G^σ is the fixed points group of σ and G_e^σ denotes its identity component.

Denote by θ the Cartan involution which commutes with σ and K the set of all fixed points of θ. Then K is a σ-stable maximal compact subgroup of G.

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Let $\mathcal{G}$ be the Lie algebra of G. The involutions of $\mathcal{G}$ induced by σ and θ are denoted by the same letters, respectively.

Suppose that $\mathcal{G} = \mathcal{H} \oplus \mathcal{Q} = \mathcal{K} \oplus \mathcal{P}$ are the decompositions of $\mathcal{G}$ into +1 and -1 eigenspaces for σ and θ, respectively, where $\mathcal{H}$ (resp. $\mathcal{K}$) is the Lie algebra of H (resp. K). Fix a maximal Abelian subspace $\mathcal{A}$ in $\mathcal{P} \cap \mathcal{Q}$ and let $\mathcal{A}^*$ denote the dual space of $\mathcal{A}$. The corresponding analytic subgroup A of $\mathcal{A}$ in G is then called the vectorial part of X. For a $\alpha \in \mathcal{A}^*$, put

$$\mathcal{G}_\alpha = \{Y \in \mathcal{G} \mid [H, Y] = \alpha(H)Y, \ \forall H \in \mathcal{A}\}.$$

Then the set $\Sigma = \{\alpha \in \mathcal{A}^* \mid \mathcal{G}_\alpha \neq \{0\}, \alpha \neq 0\}$ defines a root system with the inner product induced by the Killing form $<,>$ of $\mathcal{G}$. Moreover, the Weyl group W of Σ is defined with the normalizer $N_K(\mathcal{A})$ of $\mathcal{A}$ in K modulo the centralizer $M = Z_K(\mathcal{A})$ of $\mathcal{A}$ in K. It acts naturally on $\mathcal{A}$ and coincides via this action with the reflection group of the root system Σ.

Choose a fundamental system $\Delta = \{ \ \alpha_1, ..., \alpha_l \ \}$ of Σ, where the number l which equals dim $\mathcal{A}$ is called the split rank of the symmetric space X and denote Σ^+ the corresponding set of all positive roots in Σ. Denote by $W_{K\cap H}$ the normalizer $N_{K\cap H}(\mathcal{A})$ of $\mathcal{A}$ in K modulo the centralizer $Z_{K\cap H}(\mathcal{A})$ of $\mathcal{A}$ in K. We see that $W_{K\cap H}$ is a subgroup of W. For each element ω of W we fix a representative $\underline{\omega}$ in $N_K(\mathcal{A})$ so that $\underline{\omega} \in N_{K\cap H}(\mathcal{A})$ if $\omega \in W_{K\cap H}$.

The purpose of this note is to construct a compact real analytic manifold $\widehat{\mathbf{X}}$ in which the semisimple symmetric space $\mathbf{X} = G/H$ is realized as an open subset and that G acts analytically on it. By the Cartan decomposition $G = KAH$, we must compactify the vectorial part A. Our construction is a motivation of Oshima's construction. In [5] we proposed a construction for the Riemannian symmetric spaces G/K. Here we apply the construction for the semisimple symmetric spaces $\mathbf{X} = G/H$.

Denote by $\mathcal{G}_{\mathbf{C}}$ the complexification of $\mathcal{G}$ and $G_{\mathbf{C}}$ the corresponding analytic group. For simplicity, we assume that G is the real form of the complex Lie group $G_{\mathbf{C}}$. Let $\mathcal{A}_{\mathbf{C}}$ be the complexification of $\mathcal{A}$ and $A_{\mathbf{C}}$ be the analytic subgroup of $\mathcal{A}_{\mathbf{C}}$ in $G_{\mathbf{C}}$. For each $a \in A_{\mathbf{C}}$ and $\alpha \in \Sigma$ we define $a^\alpha = e^{\alpha . log \ a} \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ and consider the subset

$$A_{\mathbb{R}} = \{ \ a \in A_{\mathbf{C}} \mid a^\alpha \in \mathbb{R}, \ \forall \alpha \in \Sigma \ \}.$$

This note is organized as follows. In Section 1, we consider the compactification of the vectorial part based on the construction in [5]. By this way, we firstly construct an embedding of $A_{\mathbb{R}}$ into a compact real analytic manifold $\widehat{A}_{\mathbb{R}}$ which is called a compactification of $A_{\mathbb{R}}$. Then we illustrate the construction via an example. In Section 2, we construct the compact manifold $\widehat{\mathbf{X}}$ based on the action of Weyl group on $\widehat{A}_{\mathbb{R}}$ and consider the real analytic structure of $\widehat{\mathbf{X}}$ induced from the real analytic structure of $\widehat{A}_{\mathbb{R}}$.

2 Compactification of the vectorial part

Let $(\mathbf{C}^*)^\Sigma$ be the set of complexes $z = (z_\beta)_{\beta\in\Sigma}$, where $z_\beta \in \mathbf{C}^*$ and $\mathbf{C}\mathbb{P}^1$ be the 1-dimensional complex projective space. Consider the map $\varphi : A_{\mathbf{C}} \longrightarrow (\mathbf{C}^*)^\Sigma$ defined by $\varphi(a) = (a^\alpha)_{\alpha\in\Sigma},\ \forall a \in A_{\mathbf{C}}$. Then, for every $z = (z_\alpha)_{\alpha\in\Sigma} \in \varphi(A_{\mathbf{C}})$ we have

$$z_{-\alpha} = (z_\alpha)^{-1},\ \forall\alpha \in \Sigma \tag{2.1}$$

$$z_\alpha = \prod_{\gamma\in\Delta} (z_\gamma)^{k(\alpha,\gamma)},\ \forall\alpha \in \Sigma^+,\ \alpha = \sum_{\gamma\in\Delta} k(\alpha,\gamma).\gamma. \tag{2.2}$$

By using the natural embedding of $(\mathbf{C}^*)^\Sigma$ into $(\mathbf{C}\mathbb{P}^1)^\Sigma$, we get an embedding map of $A_{\mathbf{C}}$ into $(\mathbf{C}\mathbb{P}^1)^\Sigma$ denoted also by φ.

Let $\mathbf{M} = \{z \in (\mathbb{R}\mathbb{P}^1)^\Sigma \mid z_{-\alpha} = z_\alpha^{-1},\ \forall\alpha \in \Sigma\ \}$, where $\mathbb{R}\mathbb{P}^1$ is the 1-dimensional real projective space. By definition, $\mathbf{M}$ is compact. Moreover, the subset

$$\mathcal{U}_{\Sigma^+} = \left\{\ m = (m_\alpha, m_{-\alpha}) \in \mathbf{M} \mid m_\alpha \in \mathbb{R}, m_{-\alpha} \in \mathbb{R}^* \cup \{\infty\}, \forall\alpha \in \Sigma^+\right\}$$

is an open subset in $(\mathbb{R}\mathbb{P}^1)^\Sigma$ and we get a homeomorphism $\chi_{\Sigma^+} : \mathcal{U}_{\Sigma^+} \longrightarrow \mathbb{R}^{\Sigma^+}$ defined by $\chi_{\Sigma^+}(m) = (m_\alpha)_{\alpha\in\Sigma^+},\ \forall m \in \mathcal{U}_{\Sigma^+}$.

Recall that W acts on $\mathbf{M}$ by $(w.z)_\alpha = z_{w^{-1}\alpha},\ \forall\alpha \in \Sigma,\ w \in W,\ z \in \mathbf{M}$. So we have $\mathcal{U}_{w(\Sigma^+)} = w.(\mathcal{U}_{\Sigma^+}),\ \forall w \in W$. By a similar way as in [5, Lemma 1.2], we see that the pair $\{\mathcal{U}_{\Sigma^+}, \chi_{\Sigma^+}\}$ is a chart on $\mathbf{M}$ and $\{\mathcal{U}_{w(\Sigma^+)}, \chi_{w(\Sigma^+)}\}_{w\in W}$ defines an atlas of charts on $\mathbf{M}$ such that $\mathbf{M}$ becomes a real analytic submanifold.

By definition, $\varphi(A_{\mathbb{R}})$ is a subset of $(\mathbb{R}\mathbb{P}^1)^\Sigma$. Denote by $\widehat{A}_{\mathbb{R}}$ the closure of $\varphi(A_{\mathbb{R}})$ in $(\mathbb{R}\mathbb{P}^1)^\Sigma$. It follows from (2.1) and (2.2) that $\widehat{A}_{\mathbb{R}}$ is a compact subset of $\mathbf{M}$. We now define an atlas of charts on $\widehat{A}_{\mathbb{R}}$ induced from the atlas on $\mathbf{M}$.

Let $\mathcal{U}_\Delta$ be the subset of $\mathcal{U}_{\Sigma^+}$ consists of elements $m = (m_\alpha, m_{-\alpha})$ such that $m_\alpha = \prod\limits_{\gamma\in\Delta} (m_\gamma)^{k(\alpha,\gamma)},\ \forall\alpha \in \Sigma^+,\ \alpha = \sum\limits_{\gamma\in\Delta} k(\alpha,\gamma).\gamma$. Then $\mathcal{U}_\Delta$ is an open subset in $\widehat{A}_{\mathbb{R}}$. It follows that $\chi_{\Sigma^+}(\mathcal{U}_\Delta) = \{\ x \in \mathbb{R}^{\Sigma^+} \mid x_\alpha = \prod\limits_{\gamma\in\Delta} (x_\gamma)^{k(\alpha,\gamma)}\ \}$ and we get a homeomorphism $\chi_\Delta : \mathcal{U}_\Delta \longrightarrow \mathbb{R}^\Delta$ defined by $\chi_\Delta(m) = (m_\gamma)_{\gamma\in\Delta},\ \forall m \in \mathcal{U}_\Delta$. Moreover, by a similar argument as that given in [5, Theorem 1.4], we have

Theorem 2.1. *$\widehat{A}_{\mathbb{R}}$ is a compact real analytic manifold that is called a compactification of $A_{\mathbb{R}}$. The set of charts $\{\mathcal{U}_{w(\Delta)}, \chi_{w(\Delta)}\}_{w\in W}$ defines an atlas of charts on $\widehat{A}_{\mathbb{R}}$ so that the manifold $\widehat{A}_{\mathbb{R}}$ is covered by $|W|$-many charts.*

Remark 2.2. Denote by $\mathcal{G}_+ = \mathcal{G}^{\sigma\theta} = \mathcal{K} \cap \mathcal{H}\ \oplus\ \mathcal{P} \cap \mathcal{Q}$ and let

$$\Sigma_+ = \Sigma(\mathcal{A}, \mathcal{G}_+) = \{\ \alpha \in \Sigma \mid \mathcal{G}_\alpha \cap \mathcal{G}_+ \neq \{0\}, \alpha \neq 0\ \}$$

be the set of restricted roots of $\mathcal{A}$ in $\mathcal{G}_+$.

Suppose that H is connected. Since $H = (H \cap K).exp\ (\mathcal{H} \cap \mathcal{P})$ we have that $H \cap K$ is connected. Then $G_+ = (H \cap K).exp\ (\mathcal{P} \cap \mathcal{Q})$ is connected and the corresponding Weyl group $W_{K \cap H}$ of $\mathcal{A}$ in $\mathcal{G}_+$ defined by

$$W_{K\cap H} = W_{(K\cap H)_e} = W(\mathcal{A}, \mathcal{G}_+) = < s_\alpha \mid \alpha \in \Sigma_+ > .$$

Since $\mathcal{G}_+ = \mathcal{K} \cap \mathcal{H} \oplus \mathcal{P} \cap \mathcal{Q}$ is the Cartan decomposition and $K \cap H$ is a maximal compact subgroup of G_+, the coset $\mathbf{X}_+ := G_+/(K \cap H)$ becomes a Riemannian symmetric space of non-compact type. We then can apply the construction for the vectorial part of $\mathbf{X}_+$.

Denote $A_{\mathbb{R},+} = \Big\{ a \in A_{\mathbf{C}} \mid a^\gamma \in \mathbb{R},\ \forall \gamma \in \Sigma_+ \Big\}$. By construction, the corresponding compactification $\widehat{A}_{\mathbb{R},+}$ is contained in the compact manifold

$$\mathbf{M}_+ = \Big\{ z \in (I\!P^1(\mathbb{R}))^{\Sigma_+} \mid z_{-\gamma} = (z_\gamma)^{-1},\ \forall \gamma \in \Sigma_+ \Big\}.$$

Moreover, $\widehat{A}_{\mathbb{R},+}$ is a compact real analytic manifold which is covered by $|W_{K\cap H}|$-many charts. Here, each Weyl chamber of $\mathcal{A}$ for Σ_+ contains $|W_{K\cap H} \setminus W|$-corresponding Weyl chambers of $\mathcal{A}$ for Σ and these subchambers are parameterized by $W_{K\cap H} \setminus W$.

Example Consider the real semi-simple Lie group $G = SL(3, \mathbb{R})$ and denote by $\mathcal{G} = sl(3, \mathbb{R})$ the corresponding Lie algebra of G. Suppose that θ is the Cartan involution defined by $\theta(X) = ({}^tX)^{-1}$, $\forall X \in G$ and $K = SO(3, \mathbb{R})$ is the maximal compact subgroup in G with respect to θ. Then $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ is the Cartan decomposition of $\mathcal{G}$ with respect to θ, where $\mathcal{K} = SO(3, \mathbb{R})$ is the Lie algebra of K.

Let σ be the involution of G defined by

$$\sigma(X) = J\theta(X)J,\ \forall X \in G,\ \text{where } J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\mathcal{G} = \mathcal{H} \oplus \mathcal{Q}$ be the decomposition of $\mathcal{G}$ with respect to σ, where $\mathcal{H} = SO(1,2)$ is the Lie algebra of the corresponding fixed points group $H = SO_0(1,2)$.

It follows that $\mathbf{X} = G/H = SL(3, \mathbb{R})/SO_0(1,2)$ is a semisimple symmetric space of rank two and we get a maximal Abelian subspace of $\mathcal{P} \cap \mathcal{Q}$ defined by

$$\mathcal{A} = \Bigg\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \Bigg| \ t_1 + t_2 + t_3 = 0 \Bigg\}.$$

By definition, the root system Σ of $\mathcal{A}$ in $\mathcal{G}$ is $\Sigma = \{ e_i - e_j \mid 1 \leq i \neq j \leq 3 \}$ and the Weyl group W is isomorphic to S_3, the symmetric group of order three.

Moreover, the corresponding analytic subgroup in G of $\mathcal{A}$ defined by

$$A \;=\; \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \Bigm| a_1a_2a_3 \;=\; 1,\; a_i > 0 \right\} \cong\; (0,\infty)^2$$

and we get

$$A_{\mathbb{R}} \;=\; \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \Bigm| a_1a_2a_3 \;=\; 1 \right\} \cong\; (\mathbb{R}^*)^2.$$

Denote $\alpha = e_1 - e_2,\; \beta = e_2 - e_3$. We see that $\Delta = \{\alpha, \beta\}$ and $\Sigma \;=\; \Sigma^+ \cup (-\Sigma^+)$, where $\Sigma^+ = \{\; \alpha, \beta, \alpha+\beta \;\}$. Then we have

$$\mathbf{M} \;=\; \left\{ \; (z_\gamma, z_{-\gamma}) \mid z_\gamma \in \mathbb{P}^1(\mathbb{R}),\; \gamma \in \Sigma^+ \; \right\} \cong\; \mathbb{P}^1(\mathbb{R})^3.$$

It follows that the pair $\{\mathcal{U}_{\Sigma^+}, \chi_{\Sigma^+}\}$ is a chart on $\mathbf{M}$, where

$$\mathcal{U}_{\Sigma^+} = \Big\{ (m_\alpha, m_\beta, m_{\alpha+\beta}, m_{-\alpha}, m_{-\beta}, m_{-\alpha-\beta}) | m_\alpha, m_\beta, m_{\alpha+\beta} \in \mathbb{R} \; \Big\} \cong\; \mathbb{R}^3$$

is an open subset in $(\mathbb{R}\mathbb{P}^1)^{\Sigma^+}$ and $\chi_{\Sigma^+} : \mathcal{U}_{\Sigma^+} \longrightarrow \mathbb{R}^{\Sigma^+}$ is the homeomorphism defined by $\chi_{\Sigma^+}(m) = (m_\alpha, m_\beta, m_{\alpha+\beta}),\; \forall m \in \mathcal{U}_{\Sigma^+}$. Then we get

$$\mathcal{U}_\Delta = \Big\{ (m_\alpha, m_\beta, m_{\alpha+\beta}, m_{-\alpha}, m_{-\beta}, m_{-\alpha-\beta}) | m_\alpha, m_\beta, m_{\alpha+\beta} \in \mathbb{R}, m_{\alpha+\beta} = m_\alpha . m_\beta \Big\}$$

and $\quad \chi_{\Sigma^+}(\mathcal{U}_\Delta) \;=\; \{\; x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = x_1 . x_2 \;\} \;\cong\; \mathbb{R}^2$. Hence $\widehat{A}_{\mathbb{R}} \cong \mathbb{R}^2 \cup \{\infty\} \cong S^2$ is a compact smooth manifold that is covered by 6-many charts.

By definition, we see that

$$G_+ = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix} \Bigm| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL(2,\mathbb{R}),\; a^{-1} = det(A) > 0 \right\}.$$

Then $X_+ \;=\; G_+/K \cap H$ is a Riemannian symmetric space of non-compact type.

Moreover, we have $\Sigma_+ \;=\; \{\; e_i - e_j \mid 2 \leq i \neq j \leq 3 \;\} = \{\; \beta, -\beta \;\}$ and the Weyl group $W_{K\cap H}$ is isomorphic to S_2, where S_2 is the symmetric group of order two. It follows that the coset $W_{K\cap H} \setminus W$ has three elements. Note that

$$A_{\mathbb{R},+} \;=\; \Big\{ a \in A_{\mathbf{C}} |\; a^\gamma \in \mathbb{R},\; \forall \gamma \in \Sigma_+ \;\} \;\cong\; \mathbb{R}^*.$$

Then the corresponding compactification $\widehat{A}_{\mathbb{R},+}$ is contained in the compact manifold

$$\mathbf{M}_+ \;=\; \left\{ \; (z_\gamma, z_\gamma^{-1}) \mid z_\gamma \in \mathbb{P}^1(\mathbb{R}),\; \forall \gamma \in \Sigma_+ \; \right\} \cong\; \mathbb{P}^1(\mathbb{R}).$$

It follows that $\widehat{A}_{\mathbb{R},+} \cong \mathbb{R} \cup \{\infty\} \cong S^1$ is a compact smooth manifold that is covered by 2-many charts.

Here each Weyl chamber of $\mathcal{A}$ for Σ_+ contains three corresponding Weyl chambers of $\mathcal{A}$ for Σ and these subchambers are parameterized by $W_{K\cap H} \setminus W$.

3 Construction of a compact embedding

In this Section, we will construct an embedding of the semisimple symmetric space $\mathbf{X}$ into a compact real analytic manifold $\widehat{\mathbf{X}}$ such that the action of G on $\widehat{\mathbf{X}}$ is analytic. Our construction is based on the indicated construction in [5] for Riemannian symmetric spaces and similar to those of Kosters, Sekiguchi, Oshima for semisimple symmetric spaces.

First we recall some notations concerning the compactification $\widehat{A}_{\mathbb{R}}$ as in [5]. Consider the subset $\widehat{A}^-_{\mathbb{R}} = \{\ \tilde{a} \in \widehat{A}_{\mathbb{R}} \mid (\tilde{a})^\alpha \in [-1,1]\ \}$ and recall that the Weyl group W acts on $\widehat{A}_{\mathbb{R}}$ as follows $(w.\tilde{a})_\alpha = (\tilde{a})_{w^{-1}(\alpha)}$, $\forall w \in W$, $\forall \tilde{a} \in \widehat{A}_{\mathbb{R}}$. Then we have $W.\widehat{A}^-_{\mathbb{R}} = \widehat{A}_{\mathbb{R}}$ (see [5, Lemma 2.1]). Moreover, for each element $\tilde{a} \in \widehat{A}_{\mathbb{R}}$ we have a unique decomposition $\tilde{a} = (\tilde{a})_{fin}.\epsilon(\tilde{a})$, where $a_{fin} \in A_{\mathbb{R}}$ and $\epsilon(\tilde{a}) \in \widehat{A}_{\mathbb{R}}$ such that $\epsilon(\tilde{a})^\gamma \in \{\ -1, 0, +1, \infty\ \}$, $\forall \gamma \in \Delta$.

Note that $\epsilon(\tilde{a}) \in \{\ -1, 0, +1, \infty\ \}^\Delta$ and for all $\alpha = \sum_{\gamma\in\Delta} k(\alpha,\gamma).\gamma \in \Sigma$ we have

$$\epsilon(\tilde{a})^\alpha = \prod_{\gamma\in\Delta} (\epsilon(\tilde{a})^\gamma)^{|k(\alpha,\gamma)|}.$$

Motivating the Oshima's definition, $\epsilon(\tilde{a})$ is called an extended signature of the element $\tilde{a}$. Now we define parabolic subalgebras with respect to extended signatures $\epsilon(\tilde{a})$, for all $\tilde{a} \in \widehat{A}_{\mathbb{R}}$.

First we consider $\tilde{a}_t \in \widehat{A}^-_{\mathbb{R}}$ and let $F_\epsilon = \{\ \gamma \mid \epsilon_\gamma = \epsilon(\tilde{a}_t)^\gamma \neq 0\ \}$ be a subset of the simple root system Δ with respect to the extended signature $\epsilon = \epsilon(\tilde{a}_t)$. Denote $\Sigma_\epsilon = (\sum_{\gamma\in F_\epsilon} \mathbb{R}\gamma) \cap \Sigma$ and suppose that W_ϵ is the subgroup of W generated by reflections with respect to γ in F_ϵ. Let P_σ be the parabolic subgroup of G with the corresponding Langlands decomposition $P_\sigma = M_\sigma A_\sigma N_\sigma$ so that $M_\sigma A_\sigma$ is the centralizer of $\mathcal{A}$ in G and the Lie algebra $\mathcal{N}_\sigma$ of N_σ equals $\sum_{\alpha\in\Sigma^+} \mathcal{G}_\alpha$. Then we can define a parabolic subalgebra

$$\mathcal{P}_\epsilon = \mathcal{M}_\sigma + \mathcal{A}_\sigma + \sum_{\alpha\in\Sigma_\epsilon} \mathcal{G}_\alpha + \sum_{\alpha\in\Sigma^+-\Sigma_\epsilon} \mathcal{G}_\alpha$$

of $\mathcal{G}$ and its Langlands decomposition $\mathcal{P}_\epsilon = \mathcal{M}_\epsilon + \mathcal{A}_\epsilon + \mathcal{N}_\epsilon$ so that $\mathcal{A}_\epsilon \subset \mathcal{A}_\sigma$ (see [11]).

Let P_ϵ denote the corresponding parabolic subgroup of $\mathcal{P}_\epsilon$ in G. It follows that $P_\epsilon = M_\epsilon A_\epsilon N_\epsilon$ is the corresponding Langlands decomposition of P_ϵ and we define a closed subgroup $P(\epsilon)$ of G by $P(\epsilon) = (M_\epsilon \cap \underline{\omega}^{-1} H \underline{\omega}) A_\epsilon N_\epsilon$, where $\underline{\omega}$ is a representative of $\omega \in W$ in $N_K(\mathcal{A})$ so that $\underline{\omega} \in N_{K\cap H}(\mathcal{A})$ if $\omega \in W_{K\cap H}$.

In general, for each $\eta = \eta(\tilde{a})$ with $\tilde{a} = w.\tilde{a}_t \in \widehat{A}_{\mathbb{R}}$, we firstly consider the parabolic subalgebra P_ϵ with respect to the element $\epsilon = \epsilon(\tilde{a}_t)$. Then we define a parabolic subalgebra $P_\eta = w.P_\epsilon.w^{-1}$ based on the action of the Weyl group W on the parabolic subalgebra P_ϵ (see [3]).

We now define an equivalent relation on the product manifold $G \times \widehat{A}_{\mathbb{R}}$. Let $x = (g, \tilde{a})$ be an element of $G \times \widehat{A}_{\mathbb{R}}$, where $\tilde{a} = w.\tilde{a}_t$ and $t = (t_\gamma)_{\gamma\in\Delta}$, $t_\gamma \in [-1, 1]$. Then we put sgn $x = \epsilon(\tilde{a}_t) =$ sgn t, which is an element of $\{ -1, 0, 1 \}^\Delta$. Here sgn $t = (\text{sgn } t_\gamma)_{\gamma\in\Delta}$ and for an s in $\mathbb{R}$ we define sgn $s = 1$ (resp. $0, -1$) if $s > 0$ (resp. $s = 0, s < 0$).

Denote $F_x,\ \Sigma_x$ and W_x instead of $F_\epsilon,\ \Sigma_\epsilon$ and W_ϵ, respectively, we define a parabolic subalgebra

$$\mathcal{P}_x = \mathcal{M}_\sigma + \mathcal{A}_\sigma + \sum_{\alpha\in\Sigma_x} \mathcal{G}_\alpha + \sum_{\alpha\in\Sigma^+ - \Sigma_x} \mathcal{G}_\alpha$$

of $\mathcal{G}$ and its Langlands decomposition $\mathcal{P}_x = \mathcal{M}_x + \mathcal{A}_x + \mathcal{N}_x$ so that $\mathcal{A}_\epsilon \subset \mathcal{A}_\sigma$.

Moreover, denoting by P_x the corresponding subgroup of $\mathcal{P}_x$ in G, we get the Langlands decomposition $P_x = M_x A_x N_x$ and $P(x) = (M_x \cap \underline{\omega}^{-1} H \underline{\omega})) A_x N_x$ is a closed subgroup of G. Let $\{ H_1, H_2, ..., H_l \}$ denote the dual basis of $\Delta = \{ \alpha_1, ..., \alpha_l \}$, that is, $H_j \in \mathcal{A}$ and $\alpha_i(H_j) = \delta_{ij},\ \forall i, j = 1, 2, ..., l$ and put $a(x) = exp(-\sum\limits_{\gamma\in F_x} log|t_\gamma|\ H_\gamma)$, where $H_\gamma \in \{ H_1, H_2, ..., H_l \}$ with respect to γ.

Note that for all elements $x = (g, \omega.\tilde{a}_t)$ and $x' = (g', \omega'.\tilde{a}_{t'})$ of $G \times \widehat{A}_{\mathbb{R}}$ such that sgn $x =$ sgn x', we have $W_x = W_{x'}$.

Definition 3.1. We say that two elements $x = (g, \omega.\tilde{a}_t)$ and $x' = (g', \omega'.\tilde{a}_{t'})$ of $G \times \widehat{A}_{\mathbb{R}}$ are equivalent if and only if the following conditions hold:

(i) sgn $x =$ sgn x'

(ii) $W_{K\cap H}\ \omega W_x = W_{K\cap H}\ \omega' W_x$

(iii) $ga(x)P(x) = g'a(x')(M_x \cap \underline{\omega'}^{-1} H \underline{\omega}) A_x N_x$.

Remark 3.2. Let $x = (g, \omega.\tilde{a}_t)$ and $x' = (g', \omega'.\tilde{a}_{t'})$ be elements of $G \times \widehat{A}_{\mathbb{R}}$ and suppose that x and x' satisfy the above condition (iii). Then there exist $u, u' \in W_{K\cap H}$ and $v, v' \in W_x$ and $m, m' \in M$ such that $\underline{u}\ \underline{\omega}\ \underline{v}\ m = \underline{u'}\underline{\omega'}\underline{v'}m'$. Moreover, (see [11, Lemma 1.2]) the condition (iii) in Definition 2.1 is equivalent to

$$ga(x)P(x)\underline{v}m = g'a'(x')P(x')\underline{v'}m'. \tag{3.1}$$

It follows from Remark 3.2 that the Definition 3.1 really gives an equivalent relation, and we write $x \sim x'$. The quotient space of $G \times \widehat{A}_{\mathbb{R}}$ by this equivalent relation becomes a topological space with the quotient topology and is denoted by $\widehat{\mathbf{X}}$.

Let $\pi : G \times \widehat{A}_{\mathbb{R}} \longrightarrow \widehat{\mathbf{X}}$ be the natural projection. Because the action of G on $G \times \widehat{A}_{\mathbb{R}}$ are compatible with the equivalent relation, we can define an action of G on $\widehat{\mathbf{X}}$ by

$$g_1\pi(g,\tilde{a}) = \pi(g_1 g,\tilde{a}),\ \forall g, g_1 \in G, \tilde{a} \in A_{\mathbb{R}}. \tag{3.2}$$

Put $\widehat{A}_{\mathbb{R},\epsilon} = \{\ \tilde{a} \in \widehat{A}_{\mathbb{R}} \mid \epsilon(\tilde{a}) = \epsilon\ \}$ for each $\epsilon \in \{\ -1,0,1\ \}^{\Delta}$ and denote $\mathbf{X}_\epsilon = \pi(G \times \widehat{A}_{\mathbb{R},\epsilon})$. Then we have

Proposition 3.3. *(i)* $\widehat{\mathbf{X}}$ *is a compact connected* G*-space and* $\widehat{\mathbf{X}} = \bigcup_{\epsilon \in \{-1,0,1\}^{\Delta}} \mathbf{X}_\epsilon$ *gives the orbital decomposition of* $\widehat{\mathbf{X}}$ *for the action of* G *on it.*

(ii) Each $\mathbf{X}_\epsilon$ *is homeomorphic to* $G/P(\epsilon)$*. There are just* $2^{|\Delta|}$ *open orbits that are isomorphic to* G/H *and the number of compact orbits in* $\widehat{\mathbf{X}}$ *equal that of the elements of the coset* $W_{K\cap H} \setminus W$*.*

Proof (i) Since $\pi(G \times \mathcal{U}_\Delta)$ is connected and contains any open orbit, the connectedness of $\widehat{\mathbf{X}}$ is clear. Denote by $\mathcal{A}^+$ the positive chamber corresponding to Σ^+ and put $A^+ = \exp\,\mathcal{A}^+$. Let $\overline{A^+} = \{\ \exp X \mid X \in \mathcal{A}$ with $\alpha(X) \geq 0$ for all $\alpha \in \Sigma^+\ \}$ be the closure of A^+, we see that

$$\overline{A^+} = \{\ \exp\,(-\sum_\gamma (\log\,t_\gamma) H_\gamma) \mid t_\gamma \in (0,1]\ \}.$$

Consider the compact subset $K \times \widehat{A}_{\mathbb{R}} = K \times W.\widehat{A}_{\mathbb{R}}^{-} \cong K \times [-1,1]^{\Delta} \times W$ of $G \times \widehat{A}_{\mathbb{R}}$. Then the subset $\pi(K \times \widehat{A}_{\mathbb{R}})$ is also compact because it is the image of a compact set under the continuous map. Moreover, it follows from the Cartan decomposition $G = KAH$ that $G = \bigcup_{w \in W} K\overline{A^+}\underline{\omega}H$(see [11]). Hence, Definition 3.1 and Remark 3.2 prove that the compact set $\pi(K \times \widehat{A}_{\mathbb{R}})$ contains all open G-orbits in $\widehat{\mathbf{X}}$. In other words, this compact set is dense in $\widehat{\mathbf{X}}$ and therefore it must be coincided with $\widehat{\mathbf{X}}$.

(ii) Put $\tilde{a} \in \widehat{A}_{\mathbb{R},\epsilon}$ for each $\epsilon \in \{\ -1,0,1\ \}^{\Delta}$ and define a map $\Psi : G/P(\epsilon) \longrightarrow \mathbf{X}_\epsilon$ by $\Psi(gP(\epsilon)) = \pi(g,\tilde{a}),\ \forall g \in G$. Then by a similar way as in [5, Proposition 2.4], we get that the map is well defined and becomes an homeomorphism which is equivariant for the action of G.

We now define an analytic structure on $\widehat{\mathbf{X}}$ based on the analytic structure on $\widehat{A}_{\mathbb{R}}$.

Let $\mathcal{A}_{\mathcal{P}}$ be a maximal Abelian subspace of $\mathcal{P}$ containing $\mathcal{A}$ and let $\Sigma(\mathcal{A}_{\mathcal{P}},\mathcal{G})$ be the set of corresponding restricted roots. By [11, Lemma 1.4], we can assume that the representatives $\underline{\omega}$ of elements $\omega \in W$ satisfy $Ad(\underline{\omega})(\mathcal{A}_{\mathcal{P}}) = \mathcal{A}_{\mathcal{P}}$. Denote by $\mathcal{G}(\sigma)$ the reductive Lie algebra generated by $\{\mathcal{G}(\mathcal{A}_{\mathcal{P}};\lambda) \mid \lambda \in \Sigma(\mathcal{A}_{\mathcal{P}},\mathcal{G}),\ \lambda|_{\mathcal{A}} = 0\}$, where $\mathcal{G}(\mathcal{A}_{\mathcal{P}};\lambda) = \{X \in \mathcal{G} \mid [X,Y] = \lambda(Y)X \ \text{ forall } Y \in \mathcal{A}_{\mathcal{P}}\}$ and put

$$\mathcal{M}(\sigma) = \{X \in \mathcal{M}_\sigma \mid [X,Y] = \lambda(Y)X \ \text{ forall } Y \in \mathcal{G}(\sigma)\}.$$

Let $G(\sigma)$ and $M(\sigma)_0$ be the analytic subgroups of G corresponding to $\mathcal{G}(\sigma)$ and $\mathcal{M}(\sigma)$, respectively, and denote

$$M(\sigma) = M(\sigma)_0.Ad_G^{-1}(Ad(K) \cap \exp(\mathrm{ad}(\sqrt{-1}\mathcal{A}_{\mathcal{P}}))).$$

Then $G(\sigma) \subset H,\ M(\sigma) \subset N_K(\mathcal{A}_p)$ and the representative $\underline{\omega}$ normalizes $G(\sigma)$ and $M(\sigma)$ for any $\omega \in W$. Moreover, it follows from [11, Lemma 1.5] that $M_\sigma = M(\sigma)G(\sigma)$ and

$$M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1}H\underline{\omega}) \simeq M_\sigma/(M_\sigma \cap \underline{\omega}^{-1}H\underline{\omega}) \simeq M/(M \cap \underline{\omega}^{-1}H\underline{\omega}).$$

For every $g \in G$ and $w \in W$, we put $\Omega_g^w = \pi(g\overline{N}_\sigma M(\sigma) \times \mathcal{U}_{w(\Delta)})$, where $\overline{N}_\sigma$ is the analytic subgroup of G corresponding to $\overline{\mathcal{N}}_\sigma = \theta(\mathcal{N}_\sigma)$ and define a map

$$\Phi_g^w : \overline{N}_\sigma \times M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1}H\underline{\omega}) \times \mathbb{R}^\Delta \longrightarrow \Omega_g^w$$

by $\Phi_g^w(n, m, t) = \pi(gnm, w.\tilde{a}_t),\ \forall (n, m, t) \in \overline{N}_\sigma \times M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1}H\underline{\omega}) \times \mathbb{R}^\Delta$. □ By the same argument as that given in [5, Lemma 2.6], we get

Lemma 3.4. *For every $g \in G$ and $w \in W$, the map Φ_g^w is a homeomorphism of $\overline{N}_\sigma \times M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1}H\underline{\omega}) \times \mathbb{R}^\Delta$ onto an open subset $\Omega_g^w = \pi(g\overline{N}_\sigma M(\sigma) \times \mathcal{U}_{w(\Delta)})$ of $\widehat{X}$.*

For brevity, we denote $\Omega^w = \overline{N}_\sigma \times M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1}H\underline{\omega}) \times \mathbb{R}^\Delta$. Then we have

Lemma 3.5. *Let $g, g' \in G$ and $w, w' \in W$. Then the map*

$$(\Phi_{g'}^{w'})^{-1} \circ (\Phi_g^w) : (\Phi_g^w)^{-1}(\Omega_g^w \cap \Omega_{g'}^{w'}) \longrightarrow (\Phi_{g'}^{w'})^{-1}(\Omega_g^w \cap \Omega_{g'}^{w'})$$

define an analytic diffeomorphism between the open subsets of the set Ω^w.

Proof By definition, Φ_g^w is bijective and continuous. Then the map $(\Phi_{g'}^{w'})^{-1} \circ (\Phi_g^w)$ is bijective and its inverse is of the same form. So we need only to show that the map $(\Phi_{g'}^{w'})^{-1} \circ (\Phi_g^w)$ is analytic. Since $(\Phi_{g'}^{w'})^{-1} \circ (\Phi_g^w) = (\Phi_e^{w'})^{-1} \circ (\Phi_{g'^{-1}g}^w)$, we can assume that $g' = e$.

Fix an arbitrary element $u = (n_o, m_o, t_o)$ of the domain of $(\Phi_e^{w'})^{-1} \circ (\Phi_g^w)$ and put $x = (gn_o m_o, w\tilde{a}_{t_o}) \in \widehat{\mathbf{X}}$ and $u' = (n'_o, m'_o, t'_o) = ((\Phi_e^{w'})^{-1} \circ \Phi_g^w)(u)$ in $\Omega^{w'}$, we will show that the map is analytic in a neighborhood of u.

We first consider the case where $w' = w$ and $g \in \overline{N}_\sigma M(\sigma) A^x$. Here A^x is the analytic subgroup of G corresponding to $\mathcal{A}^x = \sum_{\gamma \in F_x} \mathbb{R}H_\gamma$. Suppose that $g = n_1 m_1 a_1$, where $n_1 \in \overline{\mathcal{N}}_\sigma, m_1 \in M(\sigma)$ and $a_1 \in A^x$. Then we have

$$(\Phi_e^w)^{-1} \circ (\Phi_g^w)(n, m, t) = (n_1 m_1 a_1 n (m_1 a_1)^{-1}, m_1 m, a_1 t),$$

where $a_1 t = (\exp < -a_1, \log a_1 > t_1, ..., \exp < -a_l, \log a_1 > t_l)$. It follows that the map is analytic.

Next we assume that $w' = w$ and $u' = u = (e, 1, \epsilon)$ with an $\epsilon \in \{-1, 0, 1\}^{\Delta}$. Then we have $g \in P(x)$ and [11, Lemma 1.9 (i)] there exist neighborhoods V of the origin in $\mathcal{P}(x)$ and U_0 of u in Ω^w such that for any $Y \in V$ and $s \in [0, 1]$, the map $(\Phi_e^w)^{-1} \circ \exp(sY) \circ \Phi_e^w$ defines an analytic diffeomorphism of U_0 onto a neighborhood of u. Hence we have the claim if $g \in \exp V$. Moreover, any $g \in P(x)$ can be written in the form $g = g_0 g_1 ... g_k$ with $g_0 \in M(\sigma) \cap \underline{\omega}^{-1} H \underline{\omega}$ and $g_i \in \exp V (i = 1, ..., k)$, where k is a suitable positive integer. Then we have the relation

$$(\Phi_e^w)^{-1} \circ \Phi_g^w = ((\Phi_e^w)^{-1} \circ \Phi_{g_0}^w) \circ ((\Phi_e^w)^{-1} \circ \Phi_{g_1}^w) \circ ... \circ ((\Phi_e^w)^{-1} \circ \Phi_{g_k}^w)$$

and $(\Phi_e^w)^{-1} \circ \Phi_{g_i}^w$ map the point u to the same point for $i = 0, 1, ..., k$. It follows that $(\Phi_e^w)^{-1} \circ \Phi_{g_i}^w$ are analytic in some neighborhoods of u in Ω^w and we have the claim.

We consider the case where $w' \neq w, g = e$ and $u = (e, 1, \epsilon)$. Then it follows that $u' = (e, 1, \epsilon)$ when $g' = \underline{v}' m' m^{-1} \underline{v}^{-1}$, with $v, v' \in W_x$. By a similar way as in [11, Lemma 1.9 (ii)], we can prove that the map $\Phi_{g'}^{w'} \circ (\Phi_e^w)$ is analytic in the set

$$\Omega^w(\epsilon) = \overline{N}_\sigma \times M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1} H \underline{\omega}) \times \mathbb{R}_\epsilon^\Delta,$$

where $\mathbb{R}_\epsilon^\Delta = \{t \in \mathbb{R}^\Delta \mid \text{sgn } t_\gamma = \epsilon_\gamma \text{ if } \epsilon_\gamma \neq 0\}$.

Now we consider the general case. Put $g_1 = (n_o \; \underline{m_o} \; a(t_o))^{-1}$, $g_2 = g_3^{-1} g_4^{-1} g g_1^{-1}$, where $g_3 = \underline{v}' m' m^{-1} \underline{v}^{-1}$, $g_4 = n_o' \; \underline{m_o} \; a(t_o')$ and consider the maps

$$\Phi_1 = (\Phi_e^w)^{-1} \circ \Phi_{g_1}^w, \; \Phi_2 = (\Phi_e^w)^{-1} \circ \Phi_{g_2}^w, \; \Phi_3 = (\Phi_{g_3}^{w'})^{-1} \circ (\Phi_e^w), \; \Phi_4 = (\Phi_e^{w'})^{-1} \circ \Phi_{g_4}^{w'}.$$

Then we have

$$(\Phi_e^{w'})^{-1} \circ \Phi_g^w \; = \; \Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1.$$

Since $\Phi_1(u) = (e, 1, \text{sgn } x)$ and $g_2 \in P(x)$, it follows from what we have proved that the maps Φ_1, Φ_2, Φ_3 and Φ_4 are analytic in a neighborhood of $u = (n_o, m_o, t_o)$. This implies that the map $(\Phi_e^{w'})^{-1} \circ \Phi_g^w$ is analytic in a neighborhood of u and we have the Lemma. $\Box$ Lemma 3.4 and Lemma 3.5 assures that we can define an analytic structure on $\widehat{\mathbf{X}}$ through the maps Φ_g^w so that they define analytic diffeomorphisms onto open subsets Ω_g^w of $\widehat{\mathbf{X}}$ and the action of G on $\widehat{\mathbf{X}}$ is analytic. On the other hand $\widehat{\mathbf{X}}$ is Hausdorff because Φ_g^w is homeomorphic and $\overline{N}_\sigma \times M(\sigma)/(M(\sigma) \cap \underline{\omega}^{-1} H \underline{\omega}) \times \mathbb{R}^\Delta$ is Hausdorff. Combining this with Proposition 3.3 we get

Theorem 3.6. *(i)* $\widehat{\mathbf{X}}$ *is a compact connected real analytic manifold and* $\bigcup_{w \in W, g \in G} \Omega_g^w$ *is an open covering of* $\widehat{X}$ *such that the maps* Φ_g^w *are real analytic diffeomorphisms.*

(ii) The action of G on $\widehat{\mathbf{X}}$ is analytic and the orbit $G\pi(x)$ for a point x in $\widehat{\mathbf{X}}$ is isomorphic to $G/P(x)$.

(iii) There are just $2^{|\Delta|}$ open orbits that are isomorphic to G/H and the number of compact orbits in $\widehat{\mathbf{X}}$ equal that of the elements of the coset $W_{K\cap H}\backslash W$.

Remark 3.7. (i) By a similar way, we can construct the compactification of the corresponding Riemannian symmetric spaces $\mathbf{X}_+ = G^+/K \cap H$.

(ii) For the special case $H = K$, i.e., for the case of Riemannian symmetric spaces of non-compact type, we obtain the corresponding compactification indicated in [5].

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