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Spectrum of graph parameters

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Abstract

Let $\mathcal{G}$ be the class of all graphs and $\mathcal{J} \subseteq \mathcal{G}$. A graph parameter f is called an *interpolation graph parameter with respect to* $\mathcal{J}$ if there exist integers a and b such that

$$\{f(G) : G \in \mathcal{J}\} = \{k \in \mathbb{Z} : a \leq k \leq b\}.$$

In the study of interpolation on graph parameter f with respect to $\mathcal{J}$, we may consider into two parts. First, it is to consider whether a given graph parameter f interpolates with respect to $\mathcal{J}$ or not. If it is, we shall develop techniques to find $\min(f, \mathcal{J}) := \min\{f(G) : G \in \mathcal{J}\}$ and $\max(f, \mathcal{J}) := \max\{f(G) : G \in \mathcal{J}\}$.

We discuss various kinds of graph parameters and answer the first part of interpolation theorem of graph parameters. Some of which have been done while some are new. As an application, we are able to provide an alternate proof of Erdős' conjecture on regular graphs with prescribed chromatic number.

1. Introduction

Let $\mathcal{G}$ be the class of all simple graphs. A function $f : \mathcal{G} \to \mathbb{Z}$ is called a *graph parameter* if $f(G) = f(H)$, whenever $G \cong H$. Let f be a graph parameter and $\mathcal{J} \subseteq \mathcal{G}$, f is called an *interpolation graph parameter with respect to* $\mathcal{J}$ if there exist integers a and b such that $\{f(G) : G \in \mathcal{J}\} = \{k \in \mathbb{Z} : a \leq k \leq b\}$.

In 1963, Erdős and Gallai [2] proved that any regular graph on n vertices has a chromatic number $k \leq \frac{3n}{5}$ unless the graph is complete. Erdős gave a

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conjecture that "probably such a graph exists for every $k \leq \frac{3n}{5}$, except possibly for trivial exceptional cases."

Caccetta and Pullman [1] confirmed and strengthened the above conjecture by showing that if $k > 1$, then for every $n \geq \frac{5k}{3}$, there exists a connected, regular, k-chromatic graph on n vertices. This is an example of interpolation graph parameter χ with respect to the class of all connected regular graphs of order n.

It is noted that the proof given in [1] is a constructed proof. We will show in section 5 an alternate proof of Erdős' conjecture as a consequence of our interpolation theorem of the graph parameter χ.

In 1980, G. Chartrand raised the following question: If a graph G possesses a spanning tree having m leaves and another having M leaves, where $m < M$, does G possess a spanning tree having k leaves for every k between m and M?

This question was answered affirmatively and it led to a host of lots of papers studying the interpolation properties of graph parameters with respect to the set of all spanning trees of a given graph.

In [6] - [10], Harary et. al., published some results of interpolation theorems on various kinds of graph parameters with respect to the set of all spanning trees and some classes of spanning subgraphs of a given graph.

In the study of interpolation of graph parameter f with respect to $\mathcal{J}$, we may consider into two parts. First, we consider whether a given graph parameter f interpolates with respect to $\mathcal{J}$ or not. If it is, we shall develop techniques to find

$$\min(f, \mathcal{J}) := \min\{f(G) : G \in \mathcal{J}\} \text{ and } \max(f, \mathcal{J}) := \max\{f(G) : G \in \mathcal{J}\}.$$

In this paper we present only the first part of the theorem for some graph parameters with respect to the class of all graphs with a fixed degree sequence. The second part of the theorem will also be discussed for the graph parameter χ. Finally we provide an alternate proof of Erdős' conjecture on regular graphs with prescribed chromatic number.

2. The graph of realizations

Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is called a *degree sequence* of G. Moreover, a graph H of order n is said to have the same degree sequence as G if there is a bijection $\phi\colon V(G) \to V(H)$ such that $d(v_i) = d(\phi(v_i))$ for all $i = 1, 2, \ldots, n$.

A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G and in this case, G is called a *realization* of $\mathbf{d}$.

Theorem 2.1 (Havel [11] and Hakimi [5]) *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a non-increasing sequence of non-negative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n) = \mathbf{d}'.$$

Then $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$ *is graphic.* □

A *switching* on a graph G is a replacement of any two independent edges ab and cd of G by the edges ac and bd, where ac and bd are not edges in G. It is easy to see that the graph obtained from G by a switching will have the same degree sequence as G. Thus, we may say that switching is a degree-preserving operation. It would be convenient to denote by $G^{\sigma(a,b;c,d)}$ for the graph obtained from G by the above switching.

It is easy to see that the graph obtained from G by a switching will have the same degree sequence as G. Thus, we may say that switching is a degree-preserving operation. The following theorem has been shown by Havel [11] and Hakimi [5].

Theorem 2.2 *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be a graphic degree sequence. If* G_1 *and* G_2 *are any two realizations of* $\mathbf{d}$*, then one can be obtained from the other by a finite sequence of switchings.* □

As a consequence of the above Theorem, we can define *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$, the vertices of which are the graphs with degree sequence $\mathbf{d}$; two vertices are adjacent in $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. Thus we have shown the following corollary.

Corollary 2.3 *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.* □

Let $\chi(G)$ and $\omega(G)$ be the chromatic number and the clique number of a graph G, respectively. We proved in [15] and [16] that the graph parameters χ and ω are interpolation graph parameters with respect to the set of all graphs with a fixed degree sequence. In 1956, Nordhaus and Gaddum [12] studied graph parameters $\chi+\overline{\chi}$ and $\chi\cdot\overline{\chi}$ which are defined by: $(\chi+\overline{\chi})(G) := \chi(G)+\chi(\overline{G})$ and $(\chi \cdot \overline{\chi})(G) := \chi(G) \cdot \chi(\overline{G})$, for any graph G.

Theorem 2.4 (Nordhaus and Gaddum [12]) *Let* G *be a graph on* n *vertices. Then*

(*i*) $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$,

(*ii*) $n \leq \chi(G) \cdot \chi(\overline{G}) \leq (\frac{1}{2}(n+1))^2$. □

The following result due to Fink [3] establishes the existence of a graph with prescribe chromatic number.

Theorem 2.5 (Fink [3]) *Every pair of positive integers* p *and* q *with* $p+q \leq n+1$ *and* $pq \geq n$ *there exists a graph* G *of order* n *such that* $\chi(G) = p$ *and* $\chi(\overline{G}) = q$. □

The result in Theorem 2.5 above is a kind of interpolation graph parameter $\chi + \overline{\chi}$ with respect to the set of all graphs of order n. It is easy to see that the parameter $\chi \cdot \overline{\chi}$, in general, does not have interpolation property with respect to the set of all graphs of order n.

3. Interpolation theorems

In this section, we prove interpolation theorems for three graph parameters, namely the vertex independence number, the edge independence number and the vertex domination number. Relevant graph parameters are also discussed.

A subset U of the vertex set V of a graph $G = (V, E)$ is said to be an *independent set* of G if no two vertices of U are adjacent in G. An independent set of G with maximum number of vertices is called a *maximum independent set* of G. The number of vertices in a maximum independent set of G, written $\alpha_0(G)$, is called the *independence number* of G.

A subset M of the edge set E of a graph $G = (V, E)$ is an *independent edge set* or *matching* in G if no two distinct edges in M have a common vertex. A matching M is *maximum* in G if there is no matching M' of G with $|M'| > |M|$. The cardinality of a maximum matching of G, written $\alpha_1(G)$, is called the *matching number* of G.

A vertex of a graph $G = (V, E)$ is said to cover the edges incident with it. A *vertex cover* of a graph G is a set of vertices covering all the edges of G. The minimum cardinality of a vertex cover of a graph G, written by $\beta_0(G)$, is called its *vertex covering number* of G.

An edge of a graph $G = (V, E)$ is said to cover the two vertices incident with it, and an *edge cover* of a graph G is a set of edges covering all the vertices of G. The minimum cardinality of an edge cover of G, written by $\beta_1(G)$, is called its *edge covering number* of G.

A *dominating set (*or *domset,* for short*)* of a graph $G = (V, E)$ is a subset D of V such that each vertex of $V \setminus D$ is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$ is the cardinality of a minimal dominating set with least number of elements.

Gallai ([4]) proposed a result concerning to the relationship between α_0 and β_0 as follows.

Theorem 3.1 *For any graph G of order n,* $\alpha_0 + \beta_0 = n$. □

Norman and Rabin [13] also gave the relationship between α_1, and β_1 as in Theorem 3.2.

Theorem 3.2 *For any graph G of order n and $\delta \geq 1$,* $\alpha_1 + \beta_1 = n$. □

We now prove the results on interpolation theorems on graph parameters α_0, α_1 and γ as follows:

Theorem 3.3 *If G is a graph and $\sigma(a, b; c, d)$ is a switching on G, then* $\alpha_0(G^{\sigma(a,b;c,d)}) \geq \alpha_0(G) - 1$.

Proof Let $G = (V, E)$ be a graph and let U be an independent set of vertices of V with $|U| = \alpha_0(G)$. Let $\sigma(a, b; c, d) = \sigma$ be a switching on G. Since U is an independent set of vertices, the induced subgraph of U^σ in G^σ contains at least $\alpha_0(G) - 1$ independent vertices. Therefore $\alpha_0(G^\sigma) \geq \alpha_0(G) - 1$. □

Corollary 3.4 *If σ is a switching on G, then* $|\alpha_0(G) - \alpha_0(G^\sigma)| \leq 1$.

Proof Since a switching is symmetry, we may assume that $\alpha_0(G) \geq \alpha_0(G^\sigma)$. By Theorem 3.3, $\alpha_0(G^\sigma)$ is either $\alpha_0(G) - 1$ or $\alpha_0(G)$. In both cases we have $|\alpha_0(G) - \alpha_0(G^\sigma)| \leq 1$. □

Theorem 3.5 *If G is a graph with $\alpha_1(G) = \alpha_1$ and σ is a switching on G, then $\alpha_1(G^\sigma) \geq \alpha_1 - 1$.*

Proof Let M be an independent set of edges of E with $|M| = \alpha_1(G)$. Let $\sigma(a, b; c, d) = \sigma$ be a switching on G. If $\{ab, cd\} \cap M = \emptyset$, then $|M| = |M^\sigma|$. If $\{ab, cd\} \subseteq M$, then $|M| = |M^\sigma|$. Finally, if M contains exactly one edge from the set $\{ab, cd\}$, then $|M^\sigma| = |M| - 1$. Therefore $\alpha_1(G^\sigma) \geq \alpha_1 - 1$. □

Corollary 3.6 *If σ is a switching on G, then $|\alpha_1(G) - \alpha_1(G^\sigma)| \leq 1$.*

Proof Since a switching is symmetry, we may assume that $\alpha_1(G) \geq \alpha_1(G^\sigma)$. By Theorem 3.5, $\alpha_1(G^\sigma)$ is either $\alpha_1(G) - 1$ or $\alpha_1(G)$. In both cases we have $|\alpha_1(G) - \alpha_1(G^\sigma)| \leq 1$. □

Theorem 3.7 *If G is a graph with $\gamma(G) = \gamma$ and σ is a switching on G, then $\gamma(G^\sigma) \leq \gamma + 1$.*

Proof Let D be a domset of vertices of V with $|D| = \gamma(G)$. Let $\sigma(a, b; c, d) = \sigma$ be a switching on G. If $\{a, b, c, d\} \cap D = \emptyset$, then $|D| = |D^\sigma|$. If $\{a, b, c, d\} \subseteq D$, then $|D| = |D^\sigma|$. If $a \in D$ and $b, c, d \in V \setminus D$, then $D \cup \{b\}$ is a domset of G^σ. If $a, b \in D$ and $c, d \in V \setminus D$, then D is a domset of G^σ. If $a, c \in D$, then $D \cup \{b\}$ is a domset of G^σ. Finally, if $a, b, c \in D$ and $d \in V \setminus D$, then $D \cup \{d\}$ is a domset of G^σ. Thus $\gamma(G^\sigma) \leq \gamma(G) + 1$. □

Corollary 3.8 *If σ is a switching on G, then $|\gamma(G) - \gamma(G^\sigma)| \leq 1$.*

Proof Since a switching is symmetry, we may assume that $\gamma(G) \leq \gamma(G^\sigma)$. By Theorem 3.7, $\gamma(G^\sigma)$ is either $\gamma(G) + 1$ or $\gamma(G)$. In both cases we have $|\gamma(G) - \gamma(G^\sigma)| \leq 1$. □

Combining the results in this section and with the fact that the graph of realizations is connected we can conclude the following theorem.

Theorem 3.9 *Let $\mathbf{d} = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence. Then α_0, α_1, β_0, β_1, γ are interpolation graph parameters with respect to $\mathcal{R}(\mathbf{d})$.* □

We now summarize our results for the first part of interpolation theorem as follows:

Theorem 3.10 *Let $f \in \{\chi, \omega, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then for any graphic degree sequence $\mathbf{d}$, there exist integers $a := a(f)$ and $b := b(f)$ such that $\mathbf{d}$ has a realization G with $f(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.* □

4. Application

In this final section, we first state our results in [15] and use them to proof Erdős' conjecture on regular graphs with prescribed chromatic number. Theorem 4.1 - Theorem 4.6 are cited from [15], while Theorem 4.7 - Theorem 4.8 are cited from [14].

Theorem 4.1 *If $r \geq 2$ and $n \geq 2r$, then*

$$\min(\chi,\ r^n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

□

Theorem 4.2 *If $r \geq 2$, then $\min(\chi,\ r^{r+1}) = \max(\chi,\ r^{r+1}) = r+1$, and $\min(\chi,\ r^{r+2}) = \max(\chi,\ r^{r+2}) = (r+2)/2$ (in this case r must be even).* □

Theorem 4.3 *For any $r \geq 4$ and odd integer s such that $3 \leq s \leq r$, let q and t be integers satisfying $r+s = sq+t, 0 \leq t < s$. Then*

$$\min(\chi,\ r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } 1 \leq t \leq s-2, \\ q+2 & \text{if } t = s-1. \end{cases}$$

□

Theorem 4.4 *For any even integer $r \geq 6$ and any even number s such that $4 \leq s \leq r$, let q and t be integers satisfying $r+s = sq+t,\ 0 \leq t < s$. Then*

$$\min(\chi,\ r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } t \geq 2. \end{cases}$$

□

Theorem 4.5 *Let $r \geq 2$. Then*

(1) $\max(\chi,\ r^{2r}) = r$,

(2) $\max(\chi,\ r^{2r+1}) = \begin{cases} 3 & \text{if } r = 2, \\ r & \text{if } r \geq 4, \end{cases}$

(3) $\max(\chi,\ r^n) = r+1$ for $n \geq 2r+2$. □

Theorem 4.6 *For any r and s such that $3 \leq s \leq r-1$, we have*

(1) $\max(\chi,\ r^{r+s}) \geq (r+s)/2$ *if $r+s$ is even, and*

(2) $\max(\chi,\ r^{r+s}) \geq (r+s-1)/2$ *if $r+s$ is odd.* □

We defined in [14] an $F(j)$-*graph* to be a $(j-1)$-regular graph G of minimum order $f(j)$ with $\chi(\overline{G})$ exceeds $f(j)/2$. We determined $F(j)$-graphs for all odd integers $j,\ j \geq 3$, as stated in following 2 theorems:

Theorem 4.7 *For odd integer j with $j \geq 3$, we have $f(j) = \frac{5}{2}(j-1)$ if $j \equiv 3 \pmod 4$ and $f(j) = 1 + \frac{5}{2}(j-1)$ if $j \equiv 1 \pmod 4$.* □

Theorem 4.8 *Any r-regular graph of order n with $n - r = j$ is odd and $j \geq 3$, has chromatic number at most $\dfrac{f(j)+1}{2f(j)} \cdot n$, and this bound is achieved precisely for those graphs with complement equal to a disjoint union of $F(j)$-graphs.* □

Erdős and Gallai [2] showed that any r-regular graph of order n with $r < n-1$ has a chromatic number at most $3n/5$, and this bound is achieved precisely for those graphs with complement equal to a disjoint union of 5-cycles. This means that the bound $3n/5$ is the best for an r-regular graph of order n when $r = n-3$ and the complementary graph is 2-regular. To achieve the bound, they choose the complementary graph to be the disjoint union of 5-cycles.

In the case when $j = 3$, the $F(3)$-graph is C_5 and the result of Erdős and Gallai [2] becomes a special case of Theorem 4.8.

Let n be an integer with $n \geq 4$, $r = n-3$ and $n = 5p+i,\ 0 \leq i \leq 4$. Then an explicit formula for $\max(\chi,\ r^n)$ can be given as follows.

$$\max(\chi,\ r^n) = \begin{cases} 3p & \text{if } i = 0, 1 \\ 3p+1 & \text{if } i = 2, 3, \\ 3p+2 & \text{if } i = 4. \end{cases}$$

For an integer $n \geq 4,\ n = 5p+i,\ 0 \leq i \leq 4$ and $n = 3q+t,\ 0 \leq t \leq 2$.

If n is even and $n = 2m$, then

$$\chi(m^n) = [2, m] \text{ and } \chi((n-3)^n) = [q+t, 3p+\lfloor i/2 \rfloor].$$

If n is odd, $n \geq 7$ and $n = 2m+1$, then

$$\chi(r^n) = [3, r] \text{ and } \chi((n-3)^n) = [q+t, 3p+\lfloor i/2 \rfloor],$$

where r is an even integer either $r = m-1$ or $r = m$.

It is easy to check that

$\chi(m^n) \cap \chi((n-3)^n) \neq \emptyset$ and $\chi(r^n) \cap \chi((n-3)^n) \neq \emptyset$.

Thus we have proved the following result on interpolation theorem of χ with respect to the class of all connected non complete regular graphs.

Theorem 4.9

(1) If n is even and $n \geq 6$, then there exists a connected non complete regular graph G of order n with $\chi(G) = k$ if and only if k is an integer such that $2 \leq k \leq 3n/5$.

(2) If n is odd and $n \geq 7$, then there exists a connected non complete regular graph G of order n with $\chi(G) = k$ if and only if k is an integer such that $3 \leq k \leq 3n/5$. □

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