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SYSTEM OF PARAMETERS FOR PSEUDO COHEN-MACAULAY MODULES

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1 Introduction

Throughout, $(A, \mathfrak{m})$ denotes a commutative Noetherian local ring with the maximal ideal $\mathfrak{m}$ and M a finitely generated A-module with $\dim M = d$. We denote by $Q_M(\underline{x})$ the submodule of M defined by

$$Q_M(\underline{x}) = \bigcup_{n>0} \Big((x_1^{n+1}, ..., x_d^{n+1})M : x_1^n \cdots x_d^n \Big),$$

where $\underline{x} = (x_1, ..., x_d)$ is a system of parameters on M. The submodule $Q_M(\underline{x})$ is a useful tool in the study of Monomial Conjecture, determinant maps, top local cohomology modules, modules of generalized fractions... (see [25], [4], [6] and [8]).

1.1 Definition The module M is called *pseudo Cohen-Macaulay* if there exists an system of parameters $\underline{x}$ on M such that $e(\underline{x}; M) = \ell_A(M/Q_M(\underline{x}))$

1.2 Example
If $\dim M = 1$, then M is pseudo Cohen-Macaulay by [22].

If M is Cohen-Macaulay, then one can easily see that $Q_M(\underline{x}) = (x_1, ..., x_d)M$ for an abitrary system of parameters $\underline{x} = (x_1, ..., x_d)$ of M. Thus every Cohen-Macaulay module is a pseudo Cohen-Macaulay module. The converse may be not true in general.

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This note presents some properties on systems of parameters of pseudo Cohen-Macaulay modules.

2 Preliminaries

2.1 Secondary representation, cosequences, width, Noetherian dimension of Artinian modules

Let L be an Artinian A-module with a minimal secondary representation

$$L = C_1 + \cdots + C_n,$$

where each C_i is $\mathfrak{p}_i$- secondary. The finite set $\operatorname{Att}(L) = \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ is called the set of attached prime ideals of L. Set $L_0 = \sum_{\mathfrak{p}_i \in \operatorname{Att}(L)\setminus\{\mathfrak{m}\}} C_i$. Then L_0 is independent of the choice of the minimal secondary representation of L and is called *the residuum of* L. Moreover, the length of the quotient module L/L_0 is finite. This length is called *the residual length of* L and denoted by $R\ell(L)$.

An element $a \in A$ is called *L-coregular element* if $L = aL$. The sequence of elements $a_1, ..., a_n$ of A is called *an L-cosequence* if $0 :_L (a_1, ..., a_n) \neq 0$ and a_i is $0 :_L (a_1, ..., a_{i-1})$-coregular element for every $i = 1, ..., n$. We denote by $\operatorname{Width}(L)$ the supremum of lengths of all L- cosequences in $\mathfrak{m}$.

An element $a \in \mathfrak{m}$ is called pseudo-L- coregular if $a \notin \bigcup_{\mathfrak{p} \in \operatorname{Att}(L)\setminus\{\mathfrak{m}\}} \mathfrak{p}$. Note that for each pseudo-L-coregular element $a \in \mathfrak{m}$, there exists $s \in \mathbb{N}$ such that $a^s L = L_0$.

2.1 Lemma (cf. [2, (11.3.9) and (11.3.10)]). *Let* $\mathfrak{p} \in \operatorname{Ass}(M)$. *Then,* $H_{\mathfrak{m}}^{\dim A/\mathfrak{p}}(M) \neq 0$ *and* $\mathfrak{p} \in \operatorname{Att}(H_{\mathfrak{m}}^{\dim A/\mathfrak{p}}(M))$. *Moreover,* $\operatorname{Att}(H_{\mathfrak{m}}^{d}(M)) = \{\mathfrak{p} \in \operatorname{Ass}(M) \mid \dim A/\mathfrak{p} = d\}$.

The *Noetherian dimension* of L, denoted by N- $\dim_A L$, is defined inductively as follows: when $L = 0$, put N-$\dim_A L = -1$. For an integer $d \geqslant 0$, we put N-$\dim_A L = d$ if N- $\dim_A L < d$ is false and for every ascending sequence $L_0 \subseteq L_1 \subseteq \cdots$ of submodules of L, there exists n_0 such that N-$\dim_A(L_{n+1}/L_n) < d$ for all $n > n_0$.

It is easy to see that N-$\dim_A L = 0$ if and only if L is a non-zero Noetherian module.

2.2 Lemma ([7]).

(i) *For any exact sequence of Artinian A-molules*

$$0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$$

we have N-$\dim L = \max\{N\text{-}\dim L',\ N\text{-}\dim L''\}$.

(ii) N-$\dim(L) \leqslant \dim(L)$. *The equality holds if* A *is complete.*

(iii) N-$\dim_A(L) = N$-$\dim_{\widehat{A}}(L) = \dim_{\widehat{A}}(L)$.

(iv) N-$\dim(H^i_{\mathfrak{m}}(M)) \leqslant i,\ \forall i = 0, ..., d-1$ *and* N-$\dim(H^d_{\mathfrak{m}}(M)) = d$.

2.2 The invariants $p(M), pf(M)$ and pseudo Cohen-Macaulay modules.

Let $\underline{x} = (x_1, ..., x_d)$ be an system of parameters of M and $\underline{n} = (n_1, ..., n_d)$ a d-tuple of positive integers. Set $\underline{x}(\underline{n}) = (x_1^{n_1}, ..., x_d^{n_d})A$. Consider the differences

$$I_{M,\underline{x}}(\underline{n}) = \ell\big(M/\underline{x}(\underline{n})M\big) - n_1 \dots n_d e(\underline{x}; M);$$
$$J_{M,\underline{x}}(\underline{n}) = n_1 \dots n_d \quad e(\underline{x}; M) - \ell\big(M/Q_M(\underline{x}(\underline{n})\big)$$

as functions in $n_1, \dots, n_d$, where $e(\underline{x}; M)$ is the multiplicity of M with respect to $\underline{x}$ and

$$Q_M(\underline{x}) = \bigcup_{t>0} \big((x_1^{t+1}, ..., x_d^{t+1})M : x_1^t \dots x_d^t\big).$$

In general, $I_{M,\underline{x}}(\underline{n})$ and $J_{M,\underline{x}}(\underline{n})$ are not polynomials for $n_1, \dots, n_d$ large enough (see [3], [6]). However they are bounded above by polynomials and the least degree of all polynomials in $\underline{n}$ bounding above $I_{M,\underline{x}}(\underline{n})$ (resp. $J_{M,\underline{x}}(\underline{n})$) is independent of the choice of $\underline{x}$, and it is denoted by $p(M)$ (resp. $pf(M)$). The invariant $p(M)$ is called *the polynomial type* of M (see [3]) and the invariant $pf(M)$ is called *the polynomial type of fractions* of M (see [16], [5] and [4]). For convenience we stipulate that the degree of the zero-polynomial is equal to $-\infty$. One can easy to see that following the conditions are equivalent:

(i) M is pseudo Cohen-Macaulay

(ii) $pf(M) = -\infty$

(iii) For every system of parameters $\underline{x}$ on M we have $e(\underline{x}; M) = \ell_A(M/Q_M(\underline{x}))$

Let us list basic facts on $p(M)$ and $pf(M)$ from [3], [16], and [5].

2.3 Lemma ([3] and [5]).

(i) $p(M) = p\big(M/H^0_{\mathfrak{m}}(M)\big) = p_{A/\mathrm{Ann}(M)}(M)$

$pf_A(M) = pf_A\left(M/H^0_{\mathfrak{m}}(M)\right) = pf_{A/\mathrm{Ann}(M)}(M)$

(ii) $p_A(M) = p_{\hat{A}}(\widehat{M}), pf_A(M) = pf_{\hat{A}}(\widehat{M})$, *where* $\widehat{M}$ *is the m-adic completion of* M.

(iii) *Let* $\underline{x}$ *be an system of parameters of* M *with* $\dim(0 : x_1) < d-1$. *Then*

$$pf(M/x_1M) \leqslant pf(M) \leqslant pf(M/x_1M) + 1.$$

2.4 Lemma ([5, (3.4) and (3.5)]).

(i) $p(M) \leqslant \dim M - 1$ *and if* $\dim M = d > 1$ *then* $pf(M) \leqslant d-2$.

(ii) $pf(M) \leqslant p(M)$. *If* $\mathrm{depth}(M) > p(M)$ *then* $pf(M) = p(M)$.

2.5 Lemma ([5, (3.6)]).
(i) *If* $pf(M) = -\infty$ *then* $H^i_{\mathfrak{m}}(M) = 0$ *for all* $i = p(M)+1, ..., d-1$.
(ii) *If* $pf(M) \leqslant 0$ *then* $\ell(H^i_{\mathfrak{m}}(M)) < \infty$ *for all* $i = p(M)+1, ..., d-1$.

2.6 Proposition *Assume that* $\dim M = d \geqslant 1$. *Then,*
(i) $p(M) = \max\limits_{0\leqslant i\leqslant d-1}\{N\text{-}\dim H^i_{\mathfrak{m}}(M)\}$,

(ii) *Suppose that* $p = p(M) > 0$. *Set* $\mathcal{Q} = \bigcup\limits_{i=p}^{d-1} \mathrm{Att}(H^i_{\mathfrak{m}}(M)) \setminus \{\mathfrak{m}\}$. *Let* x *be a parameter element of* M *such that* $x \notin \bigcup\limits_{\mathfrak{p}\in\mathcal{Q}} \mathfrak{p}$. *Then* $p(M/xM) = p(M) - 1$.

Proof (i). Denote $a_i(M)$ be the annihilator of the i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ of M with respect to the maximal ideal $\mathfrak{m}$ and set $\mathfrak{a}(M) = \mathfrak{a}_0(M)\cdots\mathfrak{a}_{d-1}(M)$. It follows from Lemma 2.3, [3, (3.1)] and Lemma 2.2 that

$$\begin{aligned} p_A(M) &= p_{\widehat{A}}(\widehat{M}) = \dim_{\widehat{A}} \widehat{A}/a(\widehat{M}) &&= \max_{0\leqslant i\leqslant d-1}\left\{\dim_{\widehat{A}} H^i_{\widehat{\mathfrak{m}}}(\widehat{M})\right\} \\ &= \max_{0\leqslant i\leqslant d-1}\left\{\mathrm{N\text{-}dim}_{\widehat{A}}(H^i_{\widehat{m}}(\widehat{M}))\right\} &&= \max_{0\leqslant i\leqslant d-1}\left\{\mathrm{N\text{-}dim}\,(H^i_{\mathfrak{m}}(M))\right\}. \end{aligned}$$

(ii). Let $x \notin \bigcup\limits_{\mathfrak{p}\in\mathcal{Q}} \mathfrak{p}$ be a parameter element of M. Choose $x_2, ..., x_d \in A$ such that $\underline{x} = (x, x_2, ..., x_d)$ is an system of parameters of M. For each $\mathfrak{q} \in \mathrm{Ass}(M)$ with $\dim A/\mathfrak{q} \geqslant p(M)$ we have $\mathfrak{q} \in \mathrm{Att}(H^{\dim A/\mathfrak{q}}_{\mathfrak{m}}(M))$ by Lemma 2.1. Thus $x \notin \mathfrak{q}$ for each $\mathfrak{q} \in \mathrm{Ass}(M)$ with $\dim(A/\mathfrak{q}) \geqslant p$. This implies that $\dim(0 :_M x) < p < \dim M$. Hence, $e(\underline{x}'; M/xM) = e(\underline{x}; M)$ where $\underline{x}' = (x_2, ..., x_d)$. By [3] (2.2), we get

$$I_M(\underline{n}, \underline{x}) \leqslant n_1 I_M((1, n_2, ..., n_d); \underline{x}) = n_1 I_{M/xM}((n_2, ..., n_d); \underline{x}'),$$

where $\underline{n} = (n_1, n_2, ..., n_d)$ is a d-tuple positive integers. Therefore $p(M) \leqslant p(M/xM) + 1$.

We next show the converse inequality $p(M/xM)+1 \leqslant p(M)$. As $p(M) > 0$, we need only to argue for $p(M/xM) > 0$. By the statement (i), there exists $j \in \{0, ..., d-2\}$ such that $p(M/xM) = \mathrm{N\text{-}dim}\, H^j_{\mathfrak{m}}(M/xM)$. There are only two situations arising.
<u>*Case 1:*</u> $0 \leqslant j < p$. In this case

$$p(M/xM) = \mathrm{N\text{-}dim}\,(H^j_{\mathfrak{m}}(M/xM)) \leqslant j \leqslant p-1$$

by Lemma 2.2.
<u>*Case 2:*</u> $p \leqslant j \leqslant d-2$. As $\dim(0 :_M x) < p \leqslant j$, we have $H^j_{\mathfrak{m}}(0 :_M x) = H^{j+1}_{\mathfrak{m}}(0 :_M x) = 0$. Therefore, the exact sequences

$$0 \longrightarrow (0 :_M x) \longrightarrow M \longrightarrow M/(0 :_M x) \longrightarrow 0$$

and

$$0 \longrightarrow M/(0:_M x) \stackrel{x}{\longrightarrow} M \longrightarrow M/xM \longrightarrow 0$$

induce an exact sequence of local cohomology modules

$$0 \longrightarrow H^j_{\mathfrak{m}}(M)/xH^j_m(M) \longrightarrow H^j_{\mathfrak{m}}(M/xM) \longrightarrow (0:_{H^{j+1}_{\mathfrak{m}}(M)} x) \longrightarrow 0 \qquad (1)$$

By our assumption, x is pseudo-$H^j_{\mathfrak{m}}(M)$- coregular. Consequently, $\text{N-}\dim(H^j_{\mathfrak{m}}(M)/xH^j_{\mathfrak{m}}(M)) \leqslant 0$. Moreover, when $\text{N-dim}(H^{j+1}_{\mathfrak{m}}(M)) > 0$, $x \notin \bigcup\limits_{\mathfrak{p}\in\mathcal{Q}} \mathfrak{p}$ implies that x is a parameter element on $H^{j+1}_{\mathfrak{m}}(M)$ so that

$$\text{N-}\dim(0:_{H^{j+1}_{\mathfrak{m}}(M)} x) = \text{N-}\dim(H^{j+1}_{\mathfrak{m}}(M)) - 1.$$

It now follows from the exact sequence (1) that

$$\begin{aligned} 0 &< p(M/xM) = \text{N-}\dim(H^j_{\mathfrak{m}}(M/xM)) \\ &= \max\Big\{\text{N-}\dim(H^j_{\mathfrak{m}}(M)/xH^j_m(M));\ \text{N-}\dim(0:_{H^{j+1}_{\mathfrak{m}}(M)} x)\Big\} \\ &= \text{N-}\dim(0:_{H^{j+1}_{\mathfrak{m}}(M)} x) = \text{N-}\dim(H^{j+1}_{\mathfrak{m}}(M)) - 1 \leqslant p(M) - 1. \end{aligned}$$ □

2.7 Lemma *Let $\underline{x} = (x_1, x_2, ..., x_d)$ be a system of parameters of M. Put $M_1 = M/x_1M$. For each $\underline{n}' = (n_2, ..., n_d) \in \mathbb{N}^{d-1}$, set $\underline{x}'(\underline{n}') := (x_2^{n_2}, ..., x_d^{n_d})$ and $\underline{x}(\underline{n}') = (x_1, x_2^{n_2}, ..., x_d^{n_d})$. Then, there exists an epimorphism*

$$\varphi_{\underline{n}'} : M_1/Q_{M_1}(\underline{x}'(\underline{n}')) \longrightarrow M/Q_M(\underline{x}(\underline{n}'))$$

defined by $\varphi_{\underline{n}'}\big(\overline{u} + Q_{M_1}(\underline{x}'(\underline{n}'))\big) = u + Q_M(\underline{x}(\underline{n}'))$ for each $u \in M$.

Moreover, if $x_1 \notin \bigcup\limits_{\mathfrak{q}\in \mathrm{Att}(H^{d-1}_{\mathfrak{m}}(M))\setminus\{\mathfrak{m}\}} \mathfrak{q}$, then for all $n_2, ..., n_d$ enough large, we have an exact sequence

$$0 \longrightarrow H^{d-1}_{\mathfrak{m}}(M)/x_1H^{d-1}_{\mathfrak{m}}(M) \longrightarrow M_1/Q_{M_1}(\underline{x}'(\underline{n}')) \stackrel{\varphi_{\underline{n}'}}{\longrightarrow} M/Q_M(\underline{x}(\underline{n}')) \longrightarrow 0.$$

Proof For $d-$tuples of positive integers $\underline{n} = (n_1, \ldots, n_d)$ and $\underline{m} = (m_1, \ldots, m_d)$ we define $\underline{n} \leqslant \underline{m}$ if $n_i \leqslant m_i$ for all i. Then the map

$$\delta_{\underline{n},\underline{m}} : M/Q_M(\underline{x}(\underline{n}))\longrightarrow M/Q_M(\underline{x}(\underline{m})),$$

which is defined by $\delta_{\underline{n},\underline{m}}\big(u + Q_M(\underline{x}(\underline{n}))\big) = \prod\limits_{i=1}^{d} x_i^{m_i-n_i} u + Q_M(\underline{x}(\underline{m}))$ for each $u \in M$, is injective by [4, (3.1)]. Moreover, $\{\delta_{\underline{n},\underline{m}};\ M/Q_M(\underline{x}(\underline{n}))\}$ forms a direct system and $\varinjlim\limits_{\underline{n}} M/Q_M(\underline{x}(\underline{n})) \cong H^d_{\mathfrak{m}}(M)$.

Similarly, we also have the direct system $\{\overline{\delta}_{\underline{n}',\underline{m}'};\ M_1/Q_{M_1}(\underline{x}'(\underline{n}'))\}$ and $\varinjlim_{\underline{n}'} M_1/Q_{M_1}(\underline{x}'(\underline{n}')) \cong H^{d-1}_{\mathfrak{m}}(M_1)$.

For $\underline{n}' \leqslant \underline{m}'$ with $\underline{n}' = (n_2, ..., n_d)$ and $\underline{m}' = (m_2, ..., m_d)$, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\ \varphi_{n'} & \longrightarrow & M_1/Q_{M_1}(\underline{x}'(\underline{n}')) & \xrightarrow{\varphi_{\underline{n}'}} & M/Q_M(\underline{x}(\underline{n}')) & \longrightarrow & 0 \\
 & & \downarrow \pi_{\underline{n}',\underline{m}'} & & \downarrow \overline{\delta}_{\underline{n}',\underline{m}'} & & \downarrow \delta_{\underline{n}',\underline{m}'} & & \\
0 & \longrightarrow & \mathrm{Ker}\ \varphi_{\underline{m}'} & \longrightarrow & M_1/Q_{M_1}(\underline{x}'(\underline{m}')) & \xrightarrow{\varphi_{\underline{m}'}} & M/Q_M(\underline{x}(\underline{m}')) & \longrightarrow & 0
\end{array}$$

where $\pi_{\underline{n}',\underline{m}'}$ is the reduced homomorphism. Thus, $\{\pi_{\underline{n}',\underline{m}'}; \mathrm{Ker}\ \varphi_{\underline{n}'}\}$ and $\{\delta_{\underline{n}',\underline{m}'};\ M/Q_M(\underline{x}(\underline{n}'))\}$ form direct systems. Therefore, we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\ \varphi_{n'} & \longrightarrow & M_1/Q_{M_1}(\underline{x}'(\underline{n}')) & \xrightarrow{\varphi_{\underline{n}'}} & M/Q_M(\underline{x}(\underline{n}')) & \longrightarrow & 0 \\
 & & \downarrow \pi_{\underline{n}'} & & \downarrow \overline{\delta}_{\underline{n}'} & & \downarrow \delta_{\underline{n}'} & & \\
0 & \longrightarrow & \varinjlim_{\underline{n}'} \mathrm{Ker}\ \varphi_{\underline{n}'} & \longrightarrow & H^{d-1}_{\mathfrak{m}}(M_1) & \xrightarrow{u} & \varinjlim_{\underline{n}'} M/Q_M(\underline{x}(\underline{n}')) & \longrightarrow & 0
\end{array}$$

where $\pi_{\underline{n}'}, \overline{\delta}_{\underline{n}'}$ and $\delta_{\underline{n}'}$ are the natural homomorphisms and $u = \varinjlim_{\underline{n}'} \varphi_{\underline{n}'}$. Since the homomorphisms $\delta_{\underline{n}',\underline{m}'}, \overline{\delta}_{\underline{n}',\underline{m}'}$ and $\pi_{\underline{n}',\underline{m}'}$ are injective, we have $\delta_{\underline{n}'}, \overline{\delta}_{\underline{n}'}$ and $\pi_{\underline{n}'}$ are injective.

By Lemma 2.1, $x_1 \notin \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Ass}(M)$ with $\dim A/\mathfrak{q} \geqslant d-1$. Thus $\dim(0 :_M x_1) < d-1$ and then $H^i_{\mathfrak{m}}\big(M/(0 :_M x_1)\big) \cong H^i_{\mathfrak{m}}(M)$ for $i \geqslant d-1$. Therefore, from the exact sequence

$$0 \longrightarrow M/(0 :_M x_1) \overset{x_1}{\to} \longrightarrow M \longrightarrow M_1 \longrightarrow 0,$$

we have an exact sequence of local cohomology modules

$$0 \longrightarrow H^{d-1}_{\mathfrak{m}}(M)/x_1 H^{d-1}_{\mathfrak{m}}(M) \longrightarrow H^{d-1}_{\mathfrak{m}}(M_1) \overset{\Delta}{\to} \longrightarrow H^d_{\mathfrak{m}}(M),$$

where Δ is the connecting homomorphism.

Further, we can also show a monomorphism $j : \varinjlim_{\underline{n}'} M/Q_M(\underline{x}(\underline{n}')) \longrightarrow H^d_m(M)$ such that the following diagram is commutative

$$\begin{array}{ccc} H^{d-1}_{\mathfrak{m}}(M_1) & \xrightarrow{\ \text{u}\ } & \varinjlim_{\underline{n}'} M/Q_M(\underline{x}(\underline{n}')) \\ & \Delta \searrow \quad \swarrow \text{j} & \\ & H^d_{\mathfrak{m}}(M) & \end{array}$$

Hence $\varinjlim\limits_{\underline{n}'} \operatorname{Ker} \varphi_{\underline{n}'} \cong \operatorname{Ker} u \cong \operatorname{Ker} \Delta \cong H^{d-1}_{\mathfrak{m}}(M)/x_1 H^{d-1}_{m}(M)$. Since $H^{d-1}_{\mathfrak{m}}(M)/x_1 H^{d-1}_{\mathfrak{m}}(M)$ has finite length by the choice of x_1, $\pi_{\underline{n}'}$ is an isomorphism for enough large $\underline{n}'$ ($\underline{n}' \gg 0$ for short). So we get

$$\operatorname{Ker} \varphi_{\underline{n}'} \cong H^{d-1}_{\mathfrak{m}}(M)/x_1 H^{d-1}_{m}(M)$$

for $\underline{n}' \gg 0$ as required. □

2.8 Corollary *Let M be a pseudo Cohen-Macaulay module with $p := p(M) > 0$. Let x_1 be a parameter element with $\dim(0 :_M x_1) < d-1$. Then x_1 is a $H^{d-1}_{\mathfrak{m}}(M)$-coregular element.*

Proof With the same notations and using the same argument in the proof of Lemma 2.7 we have $\varinjlim\limits_{\underline{n}'} \operatorname{Ker} \varphi_{\underline{n}'} \cong H^{d-1}_{\mathfrak{m}}(M)/x_1 H^{d-1}_{\mathfrak{m}}(M)$. On the other hand, by virtue of Lemma 2.3, M_1 is a pseudo Cohen- Macaulay module. Thus,

$$\ell_A\big(M/Q_M(\underline{x}(\underline{n}'))\big) = e(\underline{x}(\underline{n}'); M) = e(\underline{x}'(\underline{n}'); M_1) = \ell_A\big(M_1/Q_{M_1}(\underline{x}'(\underline{n}'))\big).$$

Therefore, the epimorphism $\varphi_{\underline{n}'} : M_1/Q_{M_1}(\underline{x}'(n)) \longrightarrow M/Q_M(\underline{x}(n))$ defined in Lemma 2.7 must be an isomorphism. This implies that $\operatorname{Ker} \varphi_{\underline{n}'} = 0$ for all $\underline{n}' \in \mathbb{N}^{d-1}$. Hence $\varinjlim\limits_{\underline{n}'} \operatorname{Ker} \varphi_{\underline{n}'} = 0$ and so $H^{d-1}_{\mathfrak{m}}(M) = x_1 H^{d-1}_{\mathfrak{m}}(M)$, as required. □

3 Parametric characterizations for pseudo Cohen-Macaulay modules

Following [3], a subsequence $(x_1, ..., x_j)$ of a system of parameters of M is called a reducing sequence if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}\big(M/(x_1, ..., x_{i-1})M\big)$ with $\dim A/\mathfrak{p} \geqslant d - i$, $(i = 1, ..., j)$. Note that if $\underline{x} = (x_1, ..., x_d)$ is a system of parameters on M and $x_1, ..., x_{d-1}$ form a reducing sequence, then $\underline{x}$ is just a reducing system of parameters as introduced in [1]. It should be mentioned that every A-module admits a reducing parameter system of parameters.

3.1 Definition Let $\underline{x} = (x_1, ..., x_t)$ be a sequence of elements in $\mathfrak{m}$. We set $M_i := M/(x_1, ..., x_i)M$ for all $i = 0, ..., t$. The sequence $\underline{x}$ is called *pseudo regular* for M if x_i is an $H_{\mathfrak{m}}^{d-i}(M_{i-1})$- coregular element for all $i = 1, ..., t$. If $\underline{x} = (x_1, ..., x_d)$ is an system of parameters on M and $(x_1, ..., x_{d-1})$ forms a pseudo regular sequence, then it is called a *pseudo regular system of parameters.*

3.2 Remark

(i) An abitrary system of parameters of a Cohen- Macaulay module M is a pseudo regular system of parameters

(ii) $\underline{x} = (x_1, ..., x_t)$ is a pseudo regular sequence of M if and only if $(x_1, ..., x_{j-1})$ is pseudo regular sequence of M and $(x_j, ..., x_t)$ is a pseudo regular sequence of M_{j-1} for each $j = 2, ..., t$.

(iii) By Lemma 2.1, every pseudo regular system of parameters for M is a reducing system of parameters of M.

3.3 Theorem *Assume that* $\dim M = d > 1$. *Then the following statements are equivalent:*

(i) *M is pseudo Cohen-Macaulay;*

(ii) *Any reducing system of parameters of M is pseudo regular system of parameters;*

(iii) *M admits a reducing system of parameters which is pseudo regular system of parameters;*

(iv) *M admits a pseudo regular system of parameters.*

Proof It suffices to prove that (i) $\Longrightarrow$ (ii) and (iv) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii). We prove by induction on d. Let $d = 2$ and assume that $\underline{x} = (x_1, x_2)$ is a reducing system of parameters of M. By Corollary 2.8, x_1 is $H_{\mathfrak{m}}^1(M)$-coregular and then $\underline{x}$ is pseudo regular system of parameters of M. Suppose that $d > 2$ and that our assertion is true for all pseudo Cohen- Macaulay A-modules of smaller dimension.

Let $\underline{x} = (x_1, ..., x_d)$ be a reducing system of parameters of M. As $\dim(0 : x_1) < d - 1$, then $M_1 := M/x_1M$ is pseudo Cohen-Macaulay by virtue of Lemma 2.3 (iii). The inductive hypothesis implies that $(x_2, ..., x_d)$ is a pseudo regular system of parameters of M_1.

The induction is finished now by Corollary 2.8 and Remark 3.2 (ii).

(iv) $\Longrightarrow$ (i). Again we use induction on d. Let $d = 2$, and assume that M has a pseudo regular system of parameters , say $\underline{x} = (x_1, x_2)$. Let $\underline{n} = (n_1, n_2) \in \mathbb{N}^2$. By [22, (3.2)], $J_{M,\underline{x}}(\underline{n}) = Rl(H_{\mathfrak{m}}^1(M))$ for all $n_1, n_2 \gg 0$. As x_1 is $H_{\mathfrak{m}}^1(M)$-coregular, it follows that $Rl(H_{\mathfrak{m}}^1(M)) = 0$ and so M is pseudo Cohen-Macaulay.

Assume that $d > 2$ and $\underline{x} = (x_1, x_2, ..., x_d)$ is a pseudo regular system of parameters for M. Then $\underline{x}' = (x_2, ..., x_d)$ is a pseudo regular system of parameters for $M_1 = M/x_1M$. For all $n_2, ..., n_d \geqslant 1$, set $\underline{x}(\underline{n}') = (x_1, x_2^{n_2}, ..., x_d^{n_d}), \underline{x}'(\underline{n}') = (x_2^{n_2}, ..., x_d^{n_d})$. The inductive hypothesis gives

$$\underline{x}'(\underline{n}'); M_1) = \ell\big(M_1/Q_{M_1}(\underline{x}'(\underline{n}'))\big). \tag{2}$$

On the other hand, Lemma 2.1 shows that $x_1 \notin \mathfrak{p}$, for all $\mathfrak{p} \in \operatorname{Ass}(M)$ with $\dim A/p \geqslant d-1$. We thus have $\dim(0 : x_1) < d-1$ and hence

$$e(\underline{x}'(\underline{n}'); M_1) = e(\underline{x}(\underline{n}'); M). \tag{3}$$

Take $n_2, ..., n_d$ large enough to obtain the exact sequence defined in Lemma 2.7,

$$0 \longrightarrow H^{d-1}_{\mathfrak{m}}(M)/x_1H^{d-1}_m(M) \longrightarrow M_1/Q_{M_1}(\underline{x}'(\underline{n}')) \longrightarrow M/Q_M(\underline{x}(\underline{n}')) \longrightarrow 0.$$

As x_1 is $H^{d-1}_{\mathfrak{m}}(M)$-coregular, we obtain from above exact sequence that

$$\ell_A\big(M_1/Q_{M_1}(\underline{x}'(\underline{n}'))\big) = \ell_A\big(M/Q_M(\underline{x}(\underline{n}'))\big). \tag{4}$$

Combining (2), (3) and (4), for all $n_2, ..., n_d \gg 0$, we get

$$e(\underline{x}(\underline{n}'); M) = \ell_A\big(M/Q_M(\underline{x}(\underline{n}'))\big)$$

and this finishes our proof. □

According to [9], M is called an f-module if every system of parameters $\underline{x} = (x_1, ..., x_d)$ is a M-filter regular sequence, i.e. $x_i \notin \mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Ass}(M/(x_1, ..., x_{i-1})M) \setminus \{\mathfrak{m}\}$; $(i = 1, ..., d)$.

We next combine Theorem 3.3 with [9, (2.5) and (2.11)] to obtain

3.4 Corollary *M is both f-module and pseudo Cohen-Macaulay if and only if every system of parameters for M is pseudo regular system of parameters.*

3.5 Theorem *Suppose that $p = p(M) > 0$. Then M is pseudo Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i = p+1, ..., d-1$ and there exists a subsystem of parameters $(x_1, ..., x_p)$ on M such that x_i is an $H^{p-i+1}_{\mathfrak{m}}(M_{i-1})$-coregular element for all $i = 1, ..., p$.*

Proof Assume that M is pseudo Cohen-Macaulay with $p(M) > 0$. Then $H^i_{\mathfrak{m}}(M) = 0$ for all $i = p+1, ..., d-1$ by Lemma 2.5. It follows from Corollary 2.8 that $\operatorname{Width}(H^{d-1}_{\mathfrak{m}}(M)) > 0$ and that $m \notin \operatorname{Att}(H^p_{\mathfrak{m}}(M))$. Set

$$\mathfrak{P} = \{\mathfrak{q} \in \operatorname{Ass}(M) \mid \dim A/\mathfrak{q} = d\} \ \cup \ \operatorname{Att}(H^p_{\mathfrak{m}}(M))$$

and choose $x_1 \notin \bigcup_{\mathfrak{q}\in\mathfrak{p}} \mathfrak{q}$. Obviously, x_1 is a parameter element of M and also a $H^p_{\mathfrak{m}}(M)$-coregular element. Observe that $p(M/x_1M) = p(M) - 1$ by Proposition 2.6. Now the existence of the required subsystem of parameters $(x_1, ..., x_p)$ follows by induction on p.

Conversely, assume that $H^i_{\mathfrak{m}}(M) = 0$ for all $i = p+1, ..., d-1$ and that M admits a subsystem of parameters $(x_1, ..., x_p)$ such that x_i is an $H^{p-i+1}_{\mathfrak{m}}(M)$-coregular element for all $i = 1, ..., p$. Take $x_{p+1}, ..., x_d$ such that $\underline{x} = (x_1, ..., x_p, x_{p+1}, ..., x_d)$ becomes an system of parameters on M. We will prove by induction on p that $J_{M,\underline{x}}(\underline{n}) = 0$ for all $\underline{n} \gg 0$.

The case $p = 1$ was proved in [5, (4.4)].

Assume that $p > 1$ and that our claim is true for all modules with polynomial type less than p. Set $M_1 = M/x_1M$. Because $H^i_{\mathfrak{m}}(M) = 0$ for all $i = p+1, ..., d-1$ and x_1 is $H^p_{\mathfrak{m}}(M)$-coregular, Lemma 2.1 shows that $x_1 \notin \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Ass}(M)$ with $\dim A/\mathfrak{q} \geqslant p$. Therefore $\dim(0 :_M x_1) < p \leqslant d-1, e(x_1, ..., x_d; M) = e(x_2, ..., x_d; M_1)$ and (by Proposition 2.6) $p(M_1) = p-1 > 0$. Furthermore, for all $n_2, ..., n_d \gg 0$, Lemma 2.7 gives us $M_1/Q_{M_1}(\underline{x}'(\underline{n}')) \cong M/Q_M(\underline{x}(\underline{n}'))$, where $\underline{x}'(\underline{n}') = (x_2^{n_2}, ..., x_d^{n_d})$ and $\underline{x}(\underline{n}') = (x_1, x_2^{n_2}, ..., x_d^{n_d})$. Hence

$$J_{M,\underline{x}}(\underline{n}) = J_{M_1,\underline{x}'}(\underline{n}'), \forall n_2, ..., n_d \gg 0. \tag{5}$$

On the other hand, since $\dim(0 :_M x_1) < p$ for each $i \in \{p, ..., d-1\}$, we have an exact sequence

$$0 \longrightarrow H^i_{\mathfrak{m}}(M)/x_1H^i_m(M) \longrightarrow H^i_{\mathfrak{m}}(M_1) \longrightarrow (0 :_{H^{i+1}_{\mathfrak{m}}(M)} x_1) \longrightarrow 0 \tag{6}$$

Since $H^{i+1}_{\mathfrak{m}}(M) = 0$ for all $i = p, ..., d-2$ and $\left(H^p_{\mathfrak{m}}(M)/x_1H^p_{\mathfrak{m}}(M)\right) = 0$, the exact sequence (6) implies $H^i_{\mathfrak{m}}(M_1) = 0$ for all $i = p, ..., d-2$. The induction is complete by applying the inductive hypothesis to M_1 and using the equality (5). □

The next result is an immediate consequence of Theorem 3.5 and Proposition 2.6 (ii).

3.6 Corollary *Let M be pseudo Cohen-Macaulay with $p = p(M) > 0$. Then M admits a subsystem of parameters $(x_1, ..., x_p)$ such that*

$$\mathrm{N}-\dim(H^{p-i+1}_{\mathfrak{m}}(M_{i-1})) = p-i+1$$

and

$$\mathrm{Width}(H^{p-i+1}_{\mathfrak{m}}(M_{i-1})) \geqslant \min\{2, p-i+1\}.$$

for all $i = 1, ..., p$.

The rest of this section is devoted to results on sequentially Cohen-Macaulay modules. These modules was first introduced by P. Stanley in [24] (Chapter III, 2.9) in the graded case. We recall here a definition for the local case from [8].

3.7 Definition ([8, (4.1)]. A filtration $0 = N_0 \subset N_1 \subset \ldots \subset N_t = M$ of submodules of M is said to be a *Cohen-Macaulay filtration* if

(a) Each quotient N_i/N_{i-1} is Cohen-Macaulay.

(b) $\dim N_1/N_0 < \dim N_2/N_1 < \ldots < \dim N_t/N_{t-1}$.

We say that M is *sequentially Cohen-Macaulay* if it admits a Cohen-Macaulay filtration.

3.8 Lemma *Let M be a sequentially Cohen- Macaulay A-module. Then, for all each $i = 0, ..., d$, the local cohomology module $H^i_{\mathfrak{m}}(M)$ vanishes or is a co-Cohen- Macaulay module of Noetherian dimension i.*

Proof Let $\{M_i\}_{0\leqslant i\leqslant d}$ be a Cohen-Macaulay filtration of M. Set $\mathcal{M}_i = M_i/M_{i-1}$ for all $i = 1, ..., d$ and $\mathcal{M}_0 = M_0$. If $\mathcal{M}_i$ does not vanish, then it is Cohen- Macaulay module of dimension i. It follows from [17] and [7, (3.5)] that

$$\mathrm{Width}(H^i_{\mathfrak{m}}(\mathcal{M}_i)) = i = \mathrm{N}-\dim(H^i_{\mathfrak{m}}(\mathcal{M}_i)).$$

Since $H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(\mathcal{M}_i)$, $\forall i \geqslant 0$ by [20] (5.4), this equality shows that $H^i_{\mathfrak{m}}(M)$ is a co-Cohen-Macaulay module. □

3.9 Theorem *Suppose that $d \geqslant 1$. Then the following conditions are equivalent:*

(i) *M is a sequentially Cohen-Macaulay module;*

(ii) *If $\underline{x} = (x_1, ..., x_d)$ is an abitrary filter-regular system of parameters of M then x_i is a coregular element on $H^j_{\mathfrak{m}}(M/(x_1, ..., x_{i-1})M)$ for all $j = 1, ..., d-i$ and all $i = 1, ..., d-1$;*

(iii) *There exists a filter-regular system of parameters $\underline{x} = (x_1, ..., x_d)$ of M such that x_i is a coregular element on $H^j_{\mathfrak{m}}(M/(x_1, ..., x_{i-1})M)$ for all $j = 1, ..., d-i$ and all $i = 1, ..., d-1$;*

(iv) *There exists a system of parameters $\underline{x} = (x_1, ..., x_d)$ of M such that x_i is a coregular element on $H^j_{\mathfrak{m}}(M/(x_1, ..., x_{i-1})M)$ for all $j = 1, ..., d-i$ and all $i = 1, ..., d-1$.*

Proof It is enough to prove (i) $\Longrightarrow$ (ii) and (iv) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii). We make induction on d. It is clearly true for $d = 1$. Suppose that $d \geqslant 2$ and that statement (ii) is true for all modules of dimension $< d$.

Let $\underline{x} = (x_1, ..., x_d)$ be a filter regular system of parameters of M. Let $i \in \{0, ..., d\}$. Since M is a sequentially Cohen-Macaulay, $H^i_{\mathfrak{m}}(M)$ is zero or a co-Cohen-Macaulay of Noetherian dimension i by Lemma 3.8. By Lemma 2.1 this implies that x_1 is a $H^i_{\mathfrak{m}}(M)$-coregular element. Thus $H^i_{\mathfrak{m}}(M)/x_1 H^i_{\mathfrak{m}}(M) = 0$ and $(0 :_{H^i_{\mathfrak{m}}(M)} x_1)$ is zero or a co-Cohen-Macaulay module.

On the other hand, since x_1 is a filter-regular element of M, we have $\dim(0 :_M x_1) = 0$. This yields the exact sequence

$$0 \longrightarrow H^j_{\mathfrak{m}}(M)/x_1 H^j_{\mathfrak{m}}(M) \longrightarrow H^j_{\mathfrak{m}}(M/x_1M) \longrightarrow (0 :_{H^{j+1}_{\mathfrak{m}}(M)} x_1) \longrightarrow 0$$

for all $j = 1, ..., d-2$. Thus, $H^j_{\mathfrak{m}}(M/x_1M) \cong (0 :_{H^{j+1}_{\mathfrak{m}}(M)} x_1)$ for all $j = 1, ..., d-2$. Therefore, for each $j \in \{1, ..., d\}$, $H^j_{\mathfrak{m}}(M/x_1M)$ vanishes or is a co-Cohen- Macaulay. Observe that $\underline{x}' = (x_2, ..., x_d)$ is a filter regular sequence of M/x_1M. So we get claim (ii) by induction on d.

(iv) $\Longrightarrow$ (i). We use induction on d. Clearly it is true for $d = 1$. Suppose that $d \geqslant 2$ and that statement (i) is proved for all modules of dimension $< d$. It is easy to see, that x_1 is a filter-regular element of M. Similarly as above, we obtain $H^j_{\mathfrak{m}}(M/x_1M) \cong (0 :_{H^{j+1}_{\mathfrak{m}}(M)} x_1)$ for all $j = 1, ..., d-2$. For each $j = 1, ..., d-3$, this isomorphism and the inductive hypothesis give us that $(0 :_{H^{j+1}_{\mathfrak{m}}(M)} x_1)$ is either zero a co-Cohen-Macaulay module of Noetherian dimension j.

If $(0 :_{H^{j+1}_{\mathfrak{m}}(M)} x_1) = 0$, then $H^{j+1}_{\mathfrak{m}}(M) = 0$ by the Nakayama Lemma for Artinian modules (see [12]). If $(0 :_{H^{j+1}_{\mathfrak{m}}(M)} x_1)$ is a co-Cohen-Macaulay module of Noetherian dimension j, then $H^{j+1}_{\mathfrak{m}}(M)$ is a Cohen-Macaulay module of dimension $j+1$.

As x_1 is a $H^1_{\mathfrak{m}}(M)$- coregular element, $H^1_{\mathfrak{m}}(M)$ is either zero or a co-Cohen-Macaulay module of Noetherian dimension 1. By [20, (5.5)], M is a sequentially Cohen- Macaulay module. The proof is now complete. □

3.10 Corollary *Any sequentially Cohen-Macaulay A-module is pseudo Cohen-Macaulay module.*

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