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# QUASI-DISTRIBUTIVE IMPLICATION GROUPOIDS

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**Abstract**

Distributive implication groupoids as an essential generalization of the implication reduct of intuitionistic logic were introduced and studied by the second author and I. Chajda in [3]. It has been proved that for these algebras ideals, deductive systems and congruence kernels coincide. In the paper the same connection is shown even if the implication groupoid is quasi-distributive.

## 1 Introduction

In 50-ties L. Henkin and T. Skolem introduced the notion of Hilbert algebra as an algebraic counterpart of intuitionistic logic. *A Hilbert algebra* is an algebra $\mathcal{H} = (H, \cdot, 1)$ of type (2,0) satisfying the axioms

(H1) $x \cdot (y \cdot x) = 1$
(H2) $(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1$
(H3) $x \cdot y = 1$ and $y \cdot x = 1$ imply $x = y$.

One can easily show that (H2) can be replaced by two rather simpler axioms

(LD) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$ (left distributivity)
(E) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ (exchange).

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Following [3] by an *implication groupoid* we mean any algebra $\mathcal{A} = (A, \cdot, 1)$ of type (2,0) satisfying the axioms
(IG1) $x \cdot x = 1$
(IG2) $1 \cdot x = x$.

If $\mathcal{A}$ satisfies also (LD), we call it *left distributive implication groupoid.* On each implication groupoid we can introduce the so-called *induced relation* $\leq$ by setting

$$x \leq y \text{ if and only if } x \cdot y = 1.$$

Clearly, the relation $x \leq y$ is always reflexive. In [3] it has been shown that (LD) and (E) are independent, but, on the other hand, every left distributive implication groupoid satisfies a weaker condition

(QE) $(x \cdot (y \cdot z)) \cdot (y \cdot (x \cdot z)) = 1$ (quasi-exchange)

This result immediately leads to the problem to find a weaker form of left distributivity still yielding the (QE)-property.

## 2 Quasi-distributive implication groupoids

The answer to the above question leads to the following concept:
An implication groupoid $\mathcal{A} = (A, \cdot, 1)$ satisfying the axioms
(QLD1) $(x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1$
(QLD2) $((x \cdot y) \cdot (x \cdot z)) \cdot (x \cdot (y \cdot z)) = 1$
will be called *quasi-distributive.*

Evidently, every left distributive implication groupoid is quasi-distributive. On the other hand, there are quasi-distributive groupoids not being distributive:

**Example 1** Let $\mathcal{A} = (A, \cdot, 1)$ be an implication groupoid given by the following table:

| $\cdot$ | 1 | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| 1 | 1 | $a$ | $b$ | $c$ | $d$ |
| $a$ | 1 | 1 | $b$ | $b$ | 1 |
| $b$ | 1 | $a$ | 1 | 1 | $d$ |
| $c$ | 1 | $a$ | 1 | 1 | $d$ |
| $d$ | 1 | 1 | $b$ | $c$ | 1 |

By tedious computations one can show that $\mathcal{A}$ is quasi-distributive but not distributive: we have

$$d \cdot (a \cdot c) = d \cdot b = b \neq c = 1 \cdot c = (d \cdot a) \cdot (d \cdot c)$$

and, moreover, $\mathcal{A}$ satisfies (E).

We can state several basic properties of quasi-distributive implication groupoids:

**Lemma 1** *Let $\mathcal{A} = (A, \cdot, 1)$ be a quasi-distributive implication groupoid. Then $\mathcal{A}$ satisfies the identities*
(i) $x \cdot 1 = 1$
(ii) $x \cdot (y \cdot x) = 1$
(iii) $(x \cdot (x \cdot y)) \cdot (x \cdot y) = 1$
(iv) $((x \cdot y) \cdot x) \cdot ((x \cdot y) \cdot y) = 1.$
*Moreover, the induced relation $\leq$ is the quasiorder (i.e. reflexive and transitive) and the following relationships hold:*
(v) $x \leq 1$
(vi) $x \leq y \cdot x$
(vii) $1 \leq x \Rightarrow x = 1$
(viii) $y \leq z \Rightarrow x \cdot y \leq x \cdot z$
(ix) $x \leq y \Rightarrow y \cdot z \leq x \cdot z$
(x) $x \cdot (y \cdot z) \leq y \cdot (x \cdot z)$
(xi) $x \leq (x \cdot y) \cdot y.$

**Proof** We have

$$1 = ((x \cdot x) \cdot (x \cdot x)) \cdot (x \cdot (x \cdot x)) = 1 \cdot (x \cdot 1) = x \cdot 1$$

by (QLD2), (IG1) and (IG2), hence (i) is proved.
To prove (ii), let us substitute $z = x$ in (QLD2): we get
$1 = ((x \cdot y)) \cdot (x \cdot x)) \cdot (x \cdot (y \cdot x)) = ((x \cdot y) \cdot 1) \cdot (x \cdot (y \cdot x)) = 1 \cdot ((x \cdot (y \cdot x)) = (x \cdot (y \cdot x)),$
where (IG1), (IG2) and (i) are used.
Further, putting $y = x$ and $z = y$ in (QLD1) and using (IG1) and (IG2) we obtain
$1 = (x \cdot (x \cdot y)) \cdot ((x \cdot x) \cdot (x \cdot y)) = (x \cdot (x \cdot y)) \cdot (1 \cdot (x \cdot y)) = (x \cdot (x \cdot y)) \cdot (x \cdot y).$
To get (iv), we set $x = y \cdot z$ in (QLD1):

$$1 = ((y \cdot z) \cdot (y \cdot z)) \cdot (((y \cdot z) \cdot y) \cdot ((y \cdot z) \cdot z)) = ((y \cdot z) \cdot y) \cdot ((y \cdot z) \cdot z).$$

Now we show that the relation $\leq$ is transitive. Assume $x \leq y$ and $y \leq z$ for some $x, y, z \in A$, i.e. $x \cdot y = y \cdot z = 1$. Then (QLD1) again yields

$$1 = (x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = (x \cdot 1) \cdot (1 \cdot (x \cdot z)) = x \cdot z,$$

hence $x \leq z$.
(v) and (vi) are clear from (i) and (ii), respectively.
If $1 \leq x$, then $1 = 1 \cdot x = x$, proving (vii).
To prove (viii), assume $y \leq z$. Then $y \cdot z = 1$ and by (QLD1) and (i)
$1 = (x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = (x \cdot 1) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1 \cdot ((x \cdot y) \cdot (x \cdot z)) = (x \cdot y) \cdot (x \cdot z),$

and (viii) is proved.
To prove (ix), assume $x \leq y$. Then $x \cdot y = 1$ and by (QLD1) we derive

$$x \cdot (y \cdot z) \leq (x \cdot y) \cdot (x \cdot z) = 1 \cdot (x \cdot z) = x \cdot z.$$

Using (viii) to the previous inequality we get by (ii)

$$1 = (y \cdot z) \cdot (x \cdot (y \cdot z)) \leq (y \cdot z) \cdot (x \cdot z),$$

hence $y \cdot z \leq x \cdot z$. Finally applying (ix) to the inequality $y \leq x \cdot y$ gives us

$$(x \cdot y) \cdot (x \cdot z) \leq y \cdot (x \cdot z),$$

which, by (QLD1), leads to

$$1 = (x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) \leq (x \cdot (y \cdot z)) \cdot (y \cdot (x \cdot z)),$$

hence $x \cdot (y \cdot z) \leq y \cdot (x \cdot z)$.
By (QLD2) and (iii)

$$1 = ((x \cdot (x \cdot y)) \cdot (x \cdot y)) \cdot (x \cdot ((x \cdot y) \cdot y)) = 1 \cdot (x \cdot ((x \cdot y) \cdot y)) = x \cdot ((x \cdot y) \cdot y)$$

proving $x \leq (x \cdot y) \cdot y$. □

Lemma 1 leads to the following Corollary:

**Corollary 1** *Let $\mathcal{A} = (A, \cdot, 1)$ be a quasi-distributive implication groupoid. Then $\mathcal{A}$ satisfies the (QE)-property and the induced quasiorder is an order on $A$ iff $\mathcal{A}$ is a Hilbert algebra.*

**Proof** If the induced quasiorder $\leq$ is an order relation, then $\mathcal{A}$ satisfies (LD) and (E) by Lemma 1 and hence $\mathcal{A}$ is a Hilbert algebra. The converse implication is trivial. □

The concept of implication algebra was introduced by J. C. Abbott [1] to describe properties of the logical connective implication in a classical logic. Recall that a groupoid $\mathcal{A} = (A, \cdot, 1)$ is an **implication algebra** if it satisfies the identities

(I1) $(x \cdot y) \cdot x = x$ (contraction)
(I2) $(x \cdot y) \cdot y = (y \cdot x) \cdot x$ (commutativity)
and the exchange property (E).

It is well-known that each implication groupoid satisfies also the identity $x \cdot x = y \cdot y$, i.e. $x \cdot x$ is the algebraic constant denoted by 1. The following connection between implication algebras and quasi-distributive implication groupoids is the strengthening of the main result of [6] saying that every commutative Hilbert algebra is an implication algebra:

**Theorem 1** *A quasi-distributive implication groupoid is an implication algebra iff it is commutative.*

**Proof** Let $\mathcal{A} = (A, \cdot, 1)$ be a commutative quasi-distributive implication groupoid. Let us show that the induced relation $\leq$ is an order on $A$. Indeed, assuming $x \leq y$ and $y \leq x$ we obtain

$$y = 1 \cdot y = (x \cdot y) \cdot y = (y \cdot x) \cdot x = 1 \cdot x = x.$$

By Corollary 1, $\mathcal{A}$ is a Hilbert algebra and according to [6], any commutative Hilbert algebra is an implication algebra. The converse assertion is trivial. □

# 3 Ideals, deductive systems, congruences

The concept of an ideal for Hilbert algebras coincides with that one for implication algebras and it was introduced in [3]. The concept of a deductive system for Hilbert algebras was introduced by A. Diego [4] and W. Dudek [5] proved that all these concepts coincide. We will show that the same holds even for quasi-distributive implication groupoids when the formal definitions remain unchanged:

**Definition** Let $\mathcal{A} = (A, \cdot, 1)$ be an implication groupoid. A subset $I \subseteq A$ is called an **ideal** of $\mathcal{A}$ if
(1) $1 \in I$
(2) $x \in A$ and $y \in I$ imply $x \cdot y \in I$
(3) $x \in A$ and $y_1, y_2 \in I$ imply $(y_2 \cdot (y_1 \cdot x)) \cdot x \in I$.

Let us note that if $I$ is an ideal of an implication groupoid $\mathcal{A} = (A, \cdot, 1)$ and $a \in I$ and $x \in A$, then taking $y_1 = a, y_2 = 1$ in (3) we get
(4) $(a \cdot x) \cdot x \in I$.

**Definition** Let $\mathcal{A} = (A, \cdot, 1)$ be an implication groupoid. A subset $D \subseteq A$ is called a *deductive system* of $\mathcal{A}$ if
(1) $1 \in D$
(5) $x \in D$ and $x \cdot y \in D$ imply $y \in D$.

Denote by $Id(\mathcal{A})$ or $Ded(\mathcal{A})$ the set of all ideals or the set of all deductive systems of $\mathcal{A}$, respectively.

When the binary operation "$\cdot$ " is considered to be a propositional connective implication, (5) is the expression of Modus Ponens. Thus deductive systems are just the sets of true values closed under the deductive derivation.

**Lemma 2** *Let $\mathcal{A} = (A, \cdot, 1)$ be a quasi-distributive implication groupoid. Then $Id(\mathcal{A}) = Ded(\mathcal{A})$.*

**Proof** Let $I \subseteq A$ be an ideal in $\mathcal{A}$. To prove (5), assume $x \in I$ and $x \cdot y \in I$. By (4) we know $(x \cdot y) \cdot y \in I$, hence putting $y_2 = (x \cdot y) \cdot y, y_1 = x \cdot y$ in (3) we get

$$y = 1 \cdot y = (((x \cdot y) \cdot y) \cdot ((x \cdot y) \cdot y)) \cdot y \in I.$$

Conversely, let $D$ be a deductive system of $\mathcal{A}$. Suppose $x \in A$ and $y \in D$. Then $1 = y \cdot (x \cdot y) \in D$, thus by (5) we get $x \cdot y \in D$ proving (2).
Let us prove (3). Using (QLD2) we have

$$1 = ((y \cdot (y \cdot x)) \cdot (y \cdot x)) \cdot (y \cdot ((y \cdot x) \cdot x)) \in D,$$

which by Lemma 1(iii) yields

$$y \cdot ((y \cdot x) \cdot x) = 1 \in D.$$

Applying (5) we obtain

$$(*) \qquad (y \cdot x) \cdot x \in D.$$

Assume further $y_1, y_2 \in D, x \in A$. Then according to $(*)$

$$(y_1 \cdot (y_2 \cdot x)) \cdot (y_2 \cdot x) \in D$$

and by (QE),

$$1 = ((y_1 \cdot (y_2 \cdot x)) \cdot (y_2 \cdot x)) \cdot (y_2 \cdot ((y_1 \cdot (y_2 \cdot x)) \cdot x)) \in D.$$

Finally, using (5) twice we obtain $y_2 \cdot ((y_1 \cdot (y_2 \cdot x)) \cdot x) \in D$ and
$(y_1 \cdot (y_2 \cdot x)) \cdot x \in D$. □

For an implication groupoid $\mathcal{A} = (A, \cdot, 1)$ denote by $Con\mathcal{A}$ its congruence lattice. If $\Theta \in Con\mathcal{A}$, the subset $[1]_\Theta = \{x \in A; \langle x, 1\rangle \in \Theta\}$ of $A$ is called the *congruence kernel* of $\Theta$. Denote $Ck(\mathcal{A})$ the set of all congruence kernels of $\mathcal{A}$.

**Lemma 3** *Let $\mathcal{A} = (A, \cdot, 1)$ be a quasi-distributive implication groupoid. Then Ck($\mathcal{A}$)=Id($\mathcal{A}$). Moreover, every ideal $I$ of $\mathcal{A}$ is the kernel of the congruence $\Theta_I$ defined by*

$$\langle x, y\rangle \in \Theta_I \text{ iff } x \cdot y \in I \text{ and } y \cdot x \in I,$$

*and $\Theta_I$ is the greatest congruence on $\mathcal{A}$ having the kernel $I$.*

**Proof** The inclusion $Ck(\mathcal{A}) \subseteq Id(\mathcal{A})$ holds even if $\mathcal{A}$ is an implication groupoid, see [3] .
Let us prove $Id(\mathcal{A}) \subseteq Ck(\mathcal{A})$. Since $1 \in I$ by (1), the relation $\Theta_I$ is reflexive and, evidently, it is symmetric. Now we prove transitivity of $\Theta_I$: let $\langle x, y\rangle \in \Theta_I$ and $\langle y, z\rangle \in \Theta_I$, i.e. $x \cdot y, y \cdot x, y \cdot z, z \cdot y \in I$. By (QLD1) we have

$$(**) \qquad 1 = (x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) \in I.$$

Since $y \cdot z \in I$, by (2) also $x \cdot (y \cdot z) \in I$. Moreover, $I$ is the deductive system, hence $(**)$ leads by (5) $(x \cdot y) \cdot (x \cdot z) \in I$. Applying (5) once more with respect to $x \cdot y \in I$, finally $x \cdot z \in I$.
Similarly, $z \cdot x \in I$ can be proved and $\Theta_I$ is transitive.

Let us prove the compatibility of $\Theta_I$. Assume $\langle x, y \rangle \in \Theta_I$ and $\langle u, v \rangle \in \Theta_I$, i.e. $x \cdot y, y \cdot x, u \cdot v, v \cdot u \in I$. By (QLD1) we have

$$(***) \qquad 1 = (x \cdot (u \cdot v)) \cdot ((x \cdot u) \cdot (x \cdot v)) \in I.$$

Analogously $u \cdot v \in I$ gives by (2) $x \cdot (u \cdot v) \in I$, and applying (5) with respect to $(***)$ we obtain $(x \cdot u) \cdot (x \cdot v) \in I$. Similarly $(x \cdot v) \cdot (x \cdot u) \in I$ and altogether

$$(****) \qquad \langle x \cdot u, x \cdot v \rangle \in \Theta_I.$$

Now, using the property (QE) we obtain

$$(*****) \qquad 1 = (y \cdot ((x \cdot v) \cdot v)) \cdot ((x \cdot v) \cdot (y \cdot v)) \in I.$$

According to Lemma 1 (xi) we derive $x \leq (x \cdot v) \cdot v$ and, applying (viii) of Lemma 1, $y \cdot x \leq y \cdot ((x \cdot v) \cdot v)$, thus

$$(y \cdot x) \cdot (y \cdot ((x \cdot v) \cdot v)) = 1 \in I.$$

Since $y \cdot x \in I$, by (5) also $y \cdot ((x \cdot v) \cdot v) \in I$ which, with respect to $(*****)$ and (5) again, gives

$$(x \cdot v) \cdot (y \cdot v) \in I.$$

Analogously, $(y \cdot v) \cdot (x \cdot v) \in I$ and hence $\langle x \cdot v, y \cdot v \rangle \in \Theta_I$. Finally, using $(****)$ and transitivity of $\Theta_I$, we get $\langle x \cdot u, y \cdot v \rangle \in \Theta_I$.

It is an easy exercise to show that $[1]_{\Theta_I} = I$. Assume that $\Phi$ is any congruence of $\mathcal{A}$ with the property $[1]_\Theta = I$. If $\langle x, y \rangle \in \Phi$, then

$$\langle x \cdot y, 1 \rangle = \langle x \cdot y, y \cdot y \rangle \in \Phi,$$

$$\langle y \cdot x, 1 \rangle = \langle y \cdot x, x \cdot x \rangle \in \Phi,$$

i.e. $x \cdot y, y \cdot x \in [1]_\Phi = I$. This immediately yields $\Phi \subseteq \Theta_I$ and hence $\Theta_I$ is the greatest congruence with the kernel $I$. □

Summarizing the above lemmas, we state

**Theorem 2** *Let $\mathcal{A} = (A, \cdot, 1)$ be a quasi-distributive implication groupoid. Then Id($\mathcal{A}$) = Ck($\mathcal{A}$) = Ded($\mathcal{A}$).*

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