East-West J. of Mathematics: Vol. 19, No 1 (2017) pp. 80-89

THE ANALYTICAL SOLUTION OF TWO PARAMETERS ORNSTEIN-UHLENBECK PROCESS

Pat Vatiwutipong and **Nattakorn Phewchean***

Department of Mathematics,
Faculty of Science, Mahidol University
Centre of Excellence in Mathematics, CHE
Ministry of Education, Bangkok 10400, Thailand
pat.vati@hotmail.com, nattakorn.phe@mahidol.ac.th

Abstract

The *Ornstein-Uhlenbeck process* is a well-known process which was wildly applied. This paper introduces the *two parameters Ornstein-Uhlenbeck process*, which is a simple extension of this process, by allowing two processes to depend on each other. In our research, we derive the analytical solution of our new process. Also, we investigate its properties, and give the condition for mean-reversion property. Moreover, some numerical examples are given to illustrate our result.

1. Introduction

In 1930, the *Ornstein-Uhlenbeck process* was introduced in [1] by Leonard Ornstein and George Eugene Uhlenbeck. This process is defined to be the solution of stochastic differential equation:

$$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t), \tag{1}$$

where $\theta \neq 0$, μ and $\sigma > 0$ are constant parameters and $W(t)$ is the Wiener process. The analytical solution of (1) with constant initial condition $X(0) = x_0$

*Corresponding author
Key words: Ornstein-Uhlenbeck process, stochastic differential equation, Wiener process, Gaussian and Markovian process.
2010 AMS Mathematics classification: 60H35, 34K50, 62M05.

is:

$$X(t) = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW(s), \qquad (2)$$

where $\int_0^t e^{-\theta(t-s)} dW(s)$ is a stochastic integral. The Ornstein-Uhlenbeck process is a Gaussian and Markovian process with mean $x_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$ and variance $\frac{\sigma^2}{2\theta}(1 + e^{-2\theta t})$. With positive θ, the Ornstein-Uhlenbeck process is mean-reverting, which mean that it convert to its some constant level when t tends to infinity. Its stationary (long-term) process is Gaussian with mean μ and variance $\frac{\sigma^2}{2\theta}$ (proof of these properties can be found in any standard stochastic calculus textbook such as [2]).

There are many applications of this process in several research areas. It was used to model a mean-reverting situation under the influence of friction, for example in physics, the Hookean spring whose dynamics is highly overdamped with friction coefficient (see [3]). In biology, this process was used to simulate the membrane potential of a neuron which is perturbed by electrical impulses from the surrounding network (see [4]). Moreover, this process also be applied in many financial mathematical models, for example, in [5], Oldrich Vasicek use it to model the instantaneous interest rate over times.

One of the important limitation of this process is that it depends only on itself, but in real world situation, sometimes the value that we deal with is also depends on other stochastic processes. In that case, this process cannot be applied. Many researchers tried to fix this problem by adding some stochastic parameters and got multi-parameters Ornstein-Uhlenbeck process such as [6],[7] and [8]. However, these past studies did not focus on the theoretical side and they did not consider the case that both of it depend on each other. Therefore, in this paper, we will introduce the two parameters Ornstein-Uhlenbeck process in general case, solve for its solution analytically and give a condition of mean-reverting property.

2. Preliminaries

In this section, we will introduce some well-known propositions, which can be found in [2].

Proposition 1 (Multivariate Ito's lemma) *Let $X_1(t), ..., X_n(t)$ be Ito's processes such that $dX_i(t) = a_i(t)dt + \sum_{k=1}^m b_{ik}(t)dW_k(t)$. If $f : \mathbb{R}^n \to \mathbb{R}$ is twice* *continuously differentiable, then*

$$df\big(X_1(t), ..., X_n(t)\big) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dX_i(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} d\big[X_i, X_j\big](t).$$

Proposition 2 (Zero mean property) *Let $X(t)$ be an adapted process such that $\int_0^t X^2(s)dW(s)$ is integrable. Then $\int_0^t X(s)dW(s)$ is a Gaussian process with zero mean.*

Proposition 3 (Ito's isometry property) *Let $X(t), Y(t)$ be adapted processes such that $\int_0^t X^2(s)dW_1(s)$ and $\int_0^t Y^2(s)dW_2(s)$ are integrable. Then*

$$E\left[\int_0^t X(s)dW_1(s)\int_0^t Y(s)dW_2(s)\right] = \int_0^t X(s)Y(s)d\big[W_1, W_2\big](s).$$

3. Main Result

Definition 1 The processes X_1 and X_2 are called *two parameters Ornstein-Uhlenbeck process* if it satisfy these following two stochastic differential equations:

$$\begin{aligned} dX_1(t) &= \theta_{11}\big(\mu_{11} - X_1(t)\big)dt + \theta_{12}\big(\mu_{12} - X_2(t)\big)dt + \sum_{k=1}^{m}\sigma_{1k}dW_k(t), \\ dX_2(t) &= \theta_{21}\big(\mu_{21} - X_2(t)\big)dt + \theta_{22}\big(\mu_{22} - X_1(t)\big)dt + \sum_{k=1}^{m}\sigma_{2k}dW_k(t), \end{aligned} \tag{3}$$

where σ_{ik} are not all zero for each $i = 1, 2$ and $W_1(t)$ and $W_2(t)$ are independent Wiener processes.

Lemma 1 *If $\theta_{11}\theta_{22} - \theta_{12}\theta_{21} \neq 0$, the system (3) can be rewritten as:*

$$\begin{aligned} dX_1(t) &= \theta_{11}\big(\mu_1 - X_1(t)\big)dt + \theta_{12}\big(\mu_2 - X_2(t)\big)dt + \sum_{k=1}^{m}\sigma_{1k}dW_k(t), \\ dX_2(t) &= \theta_{21}\big(\mu_1 - X_1(t)\big)dt + \theta_{22}\big(\mu_2 - X_2(t)\big)dt + \sum_{k=1}^{m}\sigma_{2k}dW_k(t). \end{aligned} \tag{4}$$

Proof. Choose μ_1 and μ_2 to be the solution of the system

$$\theta_{11}x_1 + \theta_{12}x_2 = \theta_{11}\mu_{11} + \theta_{12}\mu_{12},$$
$$\theta_{21}x_1 + \theta_{22}x_2 = \theta_{21}\mu_{21} + \theta_{22}\mu_{22},$$

which exists since $\theta_{11}\theta_{22} - \theta_{12}\theta_{21} \neq 0$. □

In real world situation, the condition $\theta_{11}\theta_{22} - \theta_{12}\theta_{21} \neq 0$ is usually hold. So it is suffices to focus on (4) instead of (3).

Theorem 1 *Consider (4) with initial condition $X_1(0) = x_0^{(1)}$ and $X_2(0) = x_0^{(2)}$. Let $\Delta = (\theta_{11} - \theta_{22})^2 + 4\theta_{12}\theta_{21}$.*

i) If $\Delta = 0$, then for $i, j \in \{1, 2\}$ with $i \neq j$,

$$X_i(t) = \mu_i + (x_0^{(i)} - \mu_i)e^{-\lambda t}(1 + (\lambda - \theta_{ii})t) - (x_0^{(j)} - \mu_j)e^{-\lambda t}\theta_{ij}t$$
$$+ \sum_{k=1}^{m} \int_0^t \sigma_{ik}e^{-\lambda(t-s)}(1 + (\lambda - \theta_{ii})(t - s)) - \sigma_{jk}e^{-\lambda(t-s)}\theta_{ij}(t - s)dW_k(s),$$

where $\lambda = \frac{1}{2}(\theta_{11} + \theta_{22})$.

ii) If $\Delta \neq 0$, then for $i, j \in \{1, 2\}$ with $i \neq j$,

$$X_i(t) = \mu_i + (x_0^{(i)} - \mu_i)\frac{(\nu - \theta_{ii})e^{-\lambda t} - (\lambda - \theta_{ii})e^{-\nu t}}{\lambda - \nu} + (x_0^{(j)} - \mu_j)\frac{\theta_{ij}(e^{-\nu t} - e^{-\lambda t})}{\lambda - \nu}$$
$$+ \sum_{k=1}^{m} \int_0^t \sigma_{ik}\frac{(\nu - \theta_{ii})e^{-\lambda(t-s)} - (\lambda - \theta_{ii})e^{-\nu(t-s)}}{\lambda - \nu}$$
$$+ \sigma_{jk}\frac{\theta_{ij}(e^{-\nu(t-s)} - e^{-\lambda(t-s)})}{\lambda - \nu}dW_k(s),$$

where $\lambda = \frac{1}{2}[(\theta_{11} + \theta_{22}) + \sqrt{\Delta}]$ and $\nu = \frac{1}{2}[(\theta_{11} + \theta_{22}) - \sqrt{\Delta}]$.

Proof Firstly, will verify that this is indeed a solution in case that $m = 1$. The result can be extend to case $m > 1$ easily.

If $\Delta = 0$, let $i, j \in \{1, 2\}$ with $i \neq j$. Define $q_{ii}(t) = e^{\lambda t}(1 - (\lambda - \theta_{11})t)$ and $q_{ij}(t) = e^{\lambda t}\theta_{ij}t$. Using Proposition 1 with functions $f_i(x_1, x_2) = q_{ii}(t)x_i + q_{ij}(t)x_j$.

Then

$$df_i(x_1, x_2) = q_{ii}(t)dx_i + q_{ij}(t)dx_j + [(q_{ii}(t)\theta_{jj} + q_{ij}(t)\theta_{ji})x_i + (q_{ii}(t)\theta_{ij} + q_{ij}(t)\theta_{jj})x_j]dt$$
$$= [(q_{ii}(t)\theta_{jj} + q_{ij}(t)\theta_{ji})\mu_i + (q_{ii}(t)\theta_{ij} + q_{ij}(t)\theta_{jj})\mu_j]dt + [q_{ii}(t)\sigma_i + q_{ij}(t)\sigma_j]dW(t)$$

Integrate both side from 0 to t and get

$$\begin{aligned}q_{ii}(t)x_i + q_{ij}(t)x_j &= x_0^{(i)} + \int_0^t (q_{ii}(t)\theta_{jj} + q_{ij}(t)\theta_{ji})\mu_i + (q_{ii}(t)\theta_{ij} + q_{ij}(t)\theta_{jj})\mu_j dt \\ &+ \int_0^t q_{ii}(t)\sigma_i + q_{ij}(t)\sigma_j dW(t) \\ &= x_0^{(i)} + (q_{ii}(t) - 1)\mu_i + q_{ij}(t)\mu_j + \int_0^t q_{ii}(t)\sigma_i + q_{ij}(t)\sigma_j dW(t)\end{aligned}$$

After that, solve the linear system for x_1 and x_2 and then the result is obtained.

On the other hand, if $\Delta \neq 0$, we prove in the similar way but using $q_{ii}(t) = \frac{(\nu-\theta_{ii})e^{\lambda t} - (\lambda+\theta_{ii})e^{\nu t}}{\lambda-\nu}$ and $q_{ij}(t) = e\frac{\theta_{ij}(e^{\nu t} - e^{\lambda t})}{\lambda-\nu}$ instead. □

Corollary 1 *The process X_1 and X_2 satisfying (4) is a Gaussian process.*

i) If $\Delta = 0$, then for $i, j \in \{1, 2\}$ with $i \neq j$,

$$\begin{aligned}E[X_i(t)] &= \mu_i + (x_0^{(i)} - \mu_i)e^{-\lambda t}(1 + (\lambda - \theta_{ii})t) - (x_0^{(j)} - \mu_j)e^{-\lambda t}\theta_{ij}t, \\ Var[X_i(t)] &= \sum_{k=1}^{m}(\frac{A_k}{2\lambda} + \frac{B_k}{4\lambda^2} + \frac{C_k}{4\lambda^3}) - [(\frac{A_k}{2\lambda} + \frac{B_k}{4\lambda^2} + \frac{C_k}{4\lambda^3}) \\ &\qquad + (\frac{B_k}{2\lambda} + \frac{C_k}{2\lambda^2})t + (\frac{C_k}{2\lambda})t^2]e^{-2\lambda t}, \\ Cov[X_i(t), X_j(t)] &= \sum_{k=1}^{m}(\frac{D_k}{2\lambda} + \frac{E_k}{4\lambda^2} + \frac{F_k}{4\lambda^3}) - [(\frac{D_k}{2\lambda} + \frac{E_k}{4\lambda^2} + \frac{F_k}{4\lambda^3}) \\ &\qquad + (\frac{E_k}{2\lambda} + \frac{F_k}{2\lambda^2})t + (\frac{F_k}{2\lambda})t^2]e^{-2\lambda t},\end{aligned}$$

where $A_k = \sigma_{ik}^2$, $B_k = 2\sigma_{ik}^2(\lambda - \theta_{ii}) - 2\sigma_{ik}\sigma_{jk}\theta_{ij}$, $C_k = \sigma_{ik}^2(\lambda - \theta_{ii})^2 + \sigma_{jk}^2\theta_{ij}^2 - 2\sigma_{ik}\sigma_{jk}\theta_{ij}(\lambda - \theta_{ii})$, $D_k = \sigma_{ik}\sigma_{jk}$, $E_k = \sigma_{ik}^2\theta_{ji} + \sigma_{jk}^2\theta_{ij}$, $F_k = \sigma_{ik}^2\theta_{ji}(\lambda - \theta_{ii}) + \sigma_{jk}^2\theta_{ij}(\lambda - \theta_{jj}) + \sigma_{ik}\sigma_{jk}((\lambda - \theta_{ii})(\lambda - \theta_{jj}) + \theta_{ij}\theta_{ji})$.

ii) If $\Delta \neq 0$, then for $i, j \in \{1, 2\}$ with $i \neq j$,

$$E[X_i(t)] = \mu_i + (x_0^{(i)} - \mu_i)\frac{(\nu - \theta_{ii})e^{-\lambda t} - (\lambda - \theta_{ii})e^{-\nu t}}{\lambda - \nu} + (x_0^{(j)} - \mu_j)\frac{\theta_{ij}(e^{-\nu t} - e^{-\lambda t})}{\lambda - \nu},$$

$$Var[X_i(t)] = \frac{1}{(\lambda - \nu)^2}\sum_{k=1}^{m}(\frac{A_k}{2\lambda} + \frac{B_k}{2(\lambda + \nu)} + \frac{C_k}{2\nu}) - (\frac{A_k}{2\lambda})e^{-2\lambda t} + (\frac{B_k}{2(\lambda + \nu)})e^{-(\lambda+\nu)t} + (\frac{C_k}{2\nu})e^{-2\nu t},$$

$$Cov[X_i(t), X_j(t)] = \frac{1}{(\lambda - \nu)^2}\sum_{k=1}^{m}(\frac{D_k}{2\lambda} + \frac{E_k}{2(\lambda + \nu)} + \frac{F_k}{2\nu}) - (\frac{D_k}{2\lambda})e^{-2\lambda t} + (\frac{E_k}{2(\lambda + \nu)})e^{-(\lambda+\nu)t} + (\frac{F_k}{2\nu})e^{-2\nu t},$$

where

$$A_k = \sigma_{ik}^2(\nu - \theta_{ii})^2 + 2\sigma_{ik}\sigma_{jk}\theta_{ij}(\nu - \theta_{ii}) + \sigma_{jk}^2\theta_{ij}^2,$$
$$B_k = -(2\sigma_{ik}^2(\nu - \theta_{ii})(\lambda - \theta_{ii}) + \sigma_{ik}\sigma_{jk}\theta_{ij}(\nu + \lambda - \theta_{ii} - \theta_{ij}) + 2\sigma_{jk}^2\theta_{ij}^2),$$
$$C_k = \sigma_{ik}^2(\lambda - \theta_{ii})^2 + 2\sigma_{ik}\sigma_{jk}\theta_{ij}(\lambda - \theta_{ii}) + \sigma_{jk}^2\theta_{ij}^2,$$
$$D_k = \sigma_{ik}^2(\nu - \theta_{ii})\theta_{ji} + \sigma_{ik}\sigma_{jk}((\nu - \theta_{ii})(\nu - \theta_{jj}) + \theta_{ij}\theta_{ji}) + \sigma_{jk}^2\theta_{ij}\theta_{ji},$$
$$E_k = -(\sigma_{ik}^2\theta_{ji}(\nu + \lambda - 2\theta_{ii}) + \sigma_{ik}\sigma_{jk}((\nu - \theta_{ii})(\lambda - \theta_{jj}) + (\lambda - \theta_{ii})(\nu - \theta_{jj}) + 2\theta_{ij}\theta_{ji}) + \sigma_{jk}^2\theta_{ij}(\nu + \lambda - 2\theta_{jj})),$$
$$F_k = \sigma_{ik}^2(\lambda - \theta_{ii})\theta_{ji} + \sigma_{ik}\sigma_{jk}((\lambda - \theta_{ii})(\lambda - \theta_{jj}) + \theta_{ij}\theta_{ji}) + \sigma_{jk}^2\theta_{ij}\theta_{ji}.$$

Proof. We can conclude by Proposition 2 that the integral term of the solution has zero mean. Then, mean of these processes is equal to the remaining terms, and hence first part of the theorem are done. Next, Proposition 3 is used to calculate the variance and covariance. In case that $\Delta = 0$,

$$\begin{aligned}
\mathrm{Var}[X_i] &= \mathrm{E}[(X_i - \mathrm{E}[X_i])^2] \\
&= \mathrm{E}[[\sum_{k=1}^{m}\int_0^t \sigma_{ik}e^{-\lambda(t-s)}(1 + (\lambda - \theta_{ii})(t - s)) + \sigma_{jk}e^{-\lambda(t-s)}\theta_{ij}(t - s)dW_k(s)]^2] \\
&= \sum_{k=1}^{m}\int_0^t [\sigma_{ik}e^{-\lambda(t-s)}(1 + (\lambda - \theta_{ii})(t - s)) + \sigma_{jk}e^{-\lambda(t-s)}\theta_{ij}(t - s)]^2 ds \\
&= \sum_{k=1}^{m}\int_0^t A_k e^{-2\lambda(t-s)} + B_k(t - s)e^{-2\lambda(t-s)} + C_k(t - s)^2 e^{-2\lambda(t-s)} ds \\
&= \sum_{k=1}^{m}(\frac{A_k}{2\lambda} + \frac{B_k}{4\lambda^2} + \frac{C_k}{4\lambda^3}) - [(\frac{A_k}{2\lambda} + \frac{B_k}{4\lambda^2} + \frac{C_k}{4\lambda^3}) + (\frac{B_k}{2\lambda} + \frac{C_k}{2\lambda^2})t + (\frac{C_k}{2\lambda})t^2]e^{-2\lambda t}.
\end{aligned}$$

The remaining cases can calculate in the similar way. □

Recall that the (one parameter) Ornstein-Uhlenbeck process is mean-reverting when $\theta > 0$ but for two parameters case is difference. We can see from the result of Theorem 1 that it depends on the sign of real part of λ and ν. Therefore, although we assume that $\theta_{ij} > 0$ for all $i, j = 1, 2$, if λ or ν have negative real part, processes may be unbounded when t tend to infinity. To get the mean-reverting processes, we need to guarantee that real part of λ and ν must be positive. The next theorem will give a sufficient condition for mean-reversion property.

Theorem 2 *The process $X_1(t)$ and $X_2(t)$ that satisfying (4) is mean-reverting if one of the following holds,*

i) $\Delta \geq 0$ and $\theta_{11}\theta_{22} - \theta_{12}\theta_{21} < 0$, or
ii) $\Delta < 0$ and $\theta_{11} + \theta_{22} > 0$, or
iii) $\theta_{11}\theta_{22} - \theta_{12}\theta_{21} < 0$ and $\theta_{11} + \theta_{22} > 0$.
where Δ was defined in Theorem 1. Moreover, if it is mean-reverting, the process $X_i(t)$ for $i = 1, 2$ will tend to be stationary Gaussian process with mean μ_i and constant variance given by $\frac{A_k}{2\lambda} + \frac{B_k}{4\lambda^2} + \frac{C_k}{4\lambda^3}$ if $\Delta = 0$ or $\frac{1}{(\lambda-\nu)^2}(\frac{A_k}{2\lambda} + \frac{B_k}{2(\lambda+\nu)} + \frac{C_k}{2\nu})$ in the other case, where A_k, B_k and C_k were defined in Corollary 1.

4. Numerical Examples

In this section, two numerical examples are given to illustrate our results.

Example 1 Let $X_1(t)$ and $X_2(t)$ be two parameters Ornstein-Uhlenbeck processes satisfying

$$dX_1(t) = (9 - X_1(t))dt - (6 - X_2(t))dt + 0.03dW_1(t) + 0.02dW_2(t),$$
$$dX_2(t) = 2(3 - X_1(t))dt + 4(-3 - X_2(t))dt + 0.01dW_2(t),$$

with initial condition $X_1(0) = X_2(0) = 0$.

Firstly, since $\theta_{11}\theta_{22} - \theta_{12}\theta_{21} \neq 0$, we solve the linear system in Lemma 1 and get $\mu_1 = 1$ and $\mu_2 = -2$. The equations are rewritten to be:

$$dX_1(t) = (1 - X_1(t))dt - (-2 - X_2(t))dt + 0.03dW_1(t) + 0.02dW_2(t),$$
$$dX_2(t) = 2(1 - X_1(t))dt + 4(-2 - X_2(t)) + 0.01dW_2(t).$$

Since this example satisfies the first condition in Theorem 2, we can conclude before solving that it is a mean-reverting process. To solve for the solution, we calculate $\lambda = 3$ and $\nu = 2$, and it follows from Theorem 1 that

$$\begin{aligned} X_1(t) &= 1 - e^{-3t} + \int_0^t 0.08e^{-2(t-s)} - 0.05e^{-3(t-s)} dW_1(s) \\ &\quad + \int_0^t 0.01e^{-2(t-s)} - 0.01e^{-3(t-s)} dW_2(s), \\ X_2(t) &= -2 + 2e^{-3t} + \int_0^t -0.06e^{-2(t-s)} + 0.06e^{-3(t-s)} dW_1(s) \\ &\quad + \int_0^t -0.05e^{-2(t-s)} + 0.06e^{-3(t-s)} dW_2(s). \end{aligned}$$

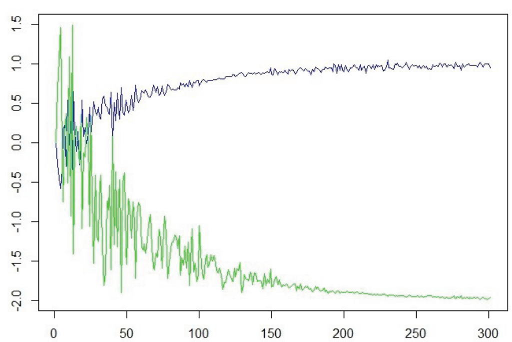

Figure 1: Simulation of Example 1

Figure 1 show the simulation of $X_1(t)$ (blue line) and $X_2(t)$ (green line) of Example 1 over the period $0 \leq t \leq 15$ (value in x-axise is $0.005t$). We can see from the graph that $X_1(t)$ and $X_2(t)$ is convert to its long-term mean which is 1 and -2 respectively, with decreasing variance converting to constants. This result is consistent with Corollary 1 and Theorem 2.

Example 2 Let $X_1(t)$ and $X_2(t)$ be two parameters Ornstein-Uhlenbeck processes satisfying

$$\begin{aligned} dX_1(t) &= 3X_1(t)dt + (1 - X_2(t))dt + 2dW_1(t) \\ dX_2(t) &= X_1(t)dt - (1 - X_2(t))dt + 5dW_1(t) \end{aligned}$$

with initial condition $X_1(0) = 2$ and $X_2(0) = -1$.

We get $\Delta = 0$ and $\lambda = -2$, so by Theorem 1,

$$X_1(t) = 2e^{2t} + 4te^{2t} + \int_0^t 2e^{2(t-s)} - 3(t-s)e^{2(t-s)} dW_1(s),$$

$$X_2(t) = 1 - 2e^{2t} + 4te^{2t} + \int_0^t 5e^{2(t-s)} - 3(t-s)e^{2(t-s)} dW_1(s).$$

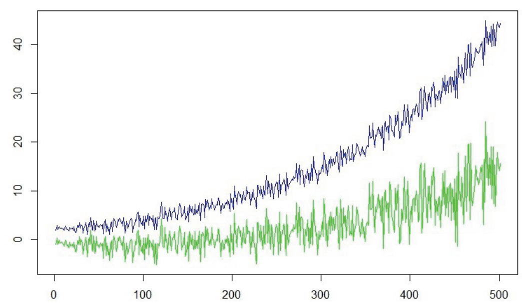

Figure 2: Simulation of Example 2

Figure 2 show the simulation of $X_1(t)$ (blue line) and $X_2(t)$ (green line) of Example 2 over the period $0 \leq t \leq 1$ (value in x-axise is $0.002t$). We can see that $X_1(t)$ and $X_2(t)$ is unbounded with increasing variance, since λ have a negative real part.

5. Conclusion

We have introduced the two parameters Ornstein-Uhlenbeck process which is an extension of the Ornstein-Uhlenbeck process by allows two processes to depend on each other. We derive its analytical solution, mean and variance Moreover, we give a mean-reverting condition and analyze its long-term behaviour. The numerical examples are given to illustrate our result. For the future work, we suggest to extend this result to be n-parameters Ornstein-Uhlenbeck process, to make it more general.

Acknowledgment This work was completed with the support of Centre of Excellence in Mathematics.

References

[1] G. E. Uhlenbeck, L. S. Ornstein, *On the theory of Brownian motion* Physical Review Letters, Physical Review, and Reviews of Modern Physics, 36, (1930) 823841

[2] F. C. Klebaner, *Introduction to Stochastic Calculus with Applications*, Imperial College Press, London, Second Edition, 2005.

[3] D.S. Lemons, "An Introduction to Stochastic Processes in Physics", Johns Hopkins University Press, Baltimore, 2002.

[4] S. Ditlevsen, A. Samson, *Introduction to Stochastic Models in Biology*, Stochastic Biomathematical Models: with Applications to Neuronal Modeling, Springer, (2013) 3-34 .

[5] O. Vasicek, *An equilibrium characterization of the term structure*, J.Financial Economics, 5, (1977) 177-188 .

[6] J. C. Hull, "Options Futures and Other Derivatives", Prentice Hall, USA, Eighth Edition, 2012.

[7] A. Lioui, *Black-Scholes-Merton revisited under stochastic dividend yields*, Journal of Futures Markets, (2006) 703-732 .

[8] N. Phewchean, Y.H. Wu, Y. Lenbury, *Option Pricing with Stochastic Volatility and Market Price of Risk: An Analytic Approach*. Recent Advances in Finite Differences and Applied & Computational Mathematics, WSEAS Proceedings, (2013) 134-139.