East-West J. of Mathematics: Vol. 21, No 1 (2019) pp. 1-19
https://doi.org/10.36853/ewjm0352

COMPLEX RINGS AND QUATERNION RINGS

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Abstract

In [4], complex rings $\boldsymbol{C}(R;-1)$, quaternion rings $\boldsymbol{H}(R;-1,-1)$ and octonion rings $\boldsymbol{O}(R;-1,-1)$ are studied for any ring R. For the real numbers $\mathbb{R}$, $\boldsymbol{C}(\mathbb{R};-1)$ is the complex numbers, $\boldsymbol{H}(\mathbb{R};-1,-1)$ is the Hamilton's quaternions and $\boldsymbol{O}(\mathbb{R};-1,-1)$ is the Cayley-Graves's octonions. In view of progress of the quaternions, generalized quaternion algebras $\left(\frac{a,b}{F}\right)$ are introduced for commutative fields F and nonzero elements $a,b \in F$, and these quaternion algebras have been extensively studied as number theory. In this paper, we use $\boldsymbol{H}(F;a,b)$ instead of $\left(\frac{a,b}{F}\right)$.

For a division ring D and nonzero elements a,b in the center of D, we introduce generalized complex rings $\boldsymbol{C}(D;a)$ and generalized quaternion rings $\boldsymbol{H}(D;a,b)$, and study the structure of these rings. We show that, if $2 \neq 0$, that is, the characteristic of D is not 2, then $\boldsymbol{H}(D;a,b)$ is a simple ring and $\boldsymbol{C}(D;a)$ is a simple ring or a direct sum of two simple

The summary of this paper was announced in Proceedings of the 50th symposium on Ring Theory and Representation Theory in Yamanashi University, Japan 2018 ([3]).
Key words: quaternion ring, complex ring, quasi-Frobenius ring, QF.
2010 AMS Classification: 11R52, 12E15, 16K20, 16L60.

rings. Main purpose of this paper is to study structures of these simple rings. We also study the case of $2 = 0$.

1 Introduction

In the middle of the 19th century, Hamilton discovered the quaternions, and Cayley and Graves independently discovered the octonions. These numbers are defined over the real numbers and contain the complex numbers. Through Frobenius, Wedderburn and many mathematicians, these numbers have been extensively studied. In particular, Frobenius showed that, up to isomorphism, the finite dimensional non-commutative division algebra over $\mathbb{R}$ is only Hamilton's quaternion algebra, and Wedderburn showed that finite division rings are commutative fields. We may say that one of roots of ring and representation theory began with these numbers.

Complex numbers, quaternion numbers and octonion numbers are naturally defined for any ring R. Let us begin to state this situation. Consider free right R-modules:

$$\begin{aligned} \boldsymbol{C}(R) &= e_0R \oplus e_1R, \\ \boldsymbol{H}(R) &= e_0R \oplus e_1R \oplus e_2R \oplus e_3R, \\ \boldsymbol{O}(R) &= e_0R \oplus e_1R \oplus \cdots \oplus e_7R. \end{aligned}$$

We define $re_i = e_ir$ for any $r \in R$ and any i $(0 \le i \le 7)$, and multiplications for $\{e_i\}_i$ are defined by the following Cayley-Graves multiplication table:

$\times$	$\boldsymbol{e}_0$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\boldsymbol{e}_4$	$\boldsymbol{e}_5$	$\boldsymbol{e}_6$	$\boldsymbol{e}_7$
$\boldsymbol{e}_0$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\boldsymbol{e}_1$	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
$\boldsymbol{e}_2$	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
$\boldsymbol{e}_3$	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
$\boldsymbol{e}_4$	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
$\boldsymbol{e}_5$	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
$\boldsymbol{e}_6$	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
$\boldsymbol{e}_7$	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Then the modules $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$ become rings, and $\boldsymbol{O}(R)$ becomes a non-associative ring. As rings, we denote these modules by $\boldsymbol{C}(R; -1)$, $\boldsymbol{H}(R; -1, -1)$ and $\boldsymbol{O}(R; -1, -1)$, and call the complex ring, quaternion ring and octonion ring, respectively. For $\boldsymbol{C}(R; -1)$ and $\boldsymbol{H}(R; -1, -1)$, we put $1 = e_0,\ i = e_1,\ j = e_2,\ k = e_3$. Then multiplications for $\{i, j, k\}$ are usual forms:

$$i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$

In order to study $\boldsymbol{H}(\boldsymbol{H}(R;-1,-1);-1,-1)$, we use $\{\mathrm{i},\mathrm{j},\mathrm{k}\}$ instead of $\{i,j,k\}$. Namely,

$$\begin{aligned}\boldsymbol{H}(R;-1,-1) &= R\oplus iR\oplus jR\oplus kR,\\ \boldsymbol{H}(\boldsymbol{H}(R;-1,-1);-1,-1) &= \boldsymbol{H}(R;-1,-1)\oplus \mathrm{i}\boldsymbol{H}(R;-1,-1)\oplus \mathrm{j}\boldsymbol{H}(R;-1,-1)\\ &\quad \oplus \mathrm{k}\boldsymbol{H}(R;-1,-1).\end{aligned}$$

Similarly, for $\boldsymbol{C}(\boldsymbol{H}(R;-1,-1);-1)$, $\boldsymbol{C}(\boldsymbol{C}(R;-1);-1)$, $\boldsymbol{H}(\boldsymbol{C}(R;-1);-1,-1)$, we use $\{\mathrm{i},\mathrm{j},\mathrm{k}\}$.

In the progress of the quaternion algebras, generalized quaternion algebras are introduced for commutative fields F. In this paper, we introduce generalized quaternion rings for any ring.

Let R be a ring and let a,b be non-zero elements in the center of R. Consider again free right R-modules $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$ above.

For these modules, we define multiplications depending on a,b. For any $r\in R$, we define

$$ri=ir,\ rj=jr,\ rk=kr$$

and define multiplications for $\{i,j,k\}$ as follows:

$$i^2=a,\ j^2=b,\ ij=-ji=k.$$

Assuming associativity, we can see the following

$$k^2=-ab,\ ik=-ki=ja,\ jk=-kj=-ib.$$

The following is the multiplication table:

$\times$	$\mathbf{1}$	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\mathbf{1}$	1	i	j	k
$\boldsymbol{i}$	i	a	k	ja
$\boldsymbol{j}$	j	$-k$	b	$-ib$
$\boldsymbol{k}$	k	$-ja$	ib	$-ab$

By these multiplications, the modules $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$ become rings. We denote these rings by $\boldsymbol{C}(R;a)$ and $\boldsymbol{H}(R;a,b)$, and call the generalized complex ring and generalized quaternion ring, respectively.

For a commutative field F, $\boldsymbol{H}(F;a,b)$ is usually denoted by $\left(\frac{a,b}{F}\right)$ and has been studied as number theory (see [6], [7], [9]).

Now, our purpose of the present paper is to study $\boldsymbol{C}(D;a)$ and $\boldsymbol{H}(D;a,b)$ for division rings D and state consistency between our theory and classical theory.

Now, let R be a ring. We denote its Jacobson radical, right socle and left socle by $J(R)$, $S(R_R)$ and $S({}_RR)$, respectively. When $S(R_R)=S({}_RR)$, we simply denote it by $S(R)$. $Z(R)$ denotes the center of R, and $M_n(R)$ denotes the $n\times n$

matrix ring over R. For a semiperfect ring R, $Pi(R)$ denotes a complete set of orthogonal primitive idempotents. Its cardinal number is uniquely determined and is denoted by $|Pi(R)|$. Furthermore we use the following symbols:

$\mathbb{R}$	the real numbers
$\mathbb{Q}$	the rational numbers
$\mathbb{C}$	the complex numbers
$ch(R)$	the characteristic of a ring R
F	commutative field
D	division ring
$\sqrt{a}$	square root of $a \in D$, i.e. $\sqrt{a}$ in D and $(\sqrt{a})^2 = a$

2 Structure of $\boldsymbol{H}(D;-1,-1)$

Recently, Lee-Oshiro [4] showed the following results.

Theorem A. *If R is a Frobenius algebra, then $\boldsymbol{C}(R;-1)$, $\boldsymbol{H}(R;-1,-1)$ and $\boldsymbol{O}(R;-1,-1)$ are Frobenius algebras.*

Theorem B. *If R is a quasi-Frobenius ring, then $\boldsymbol{C}(R;-1)$ and $\boldsymbol{H}(R;-1,-1)$ are quasi-Frobenius rings. In the case $2=0$, that is, $ch(R)=2$, $\boldsymbol{O}(R;-1,-1)$ is also a quasi-Frobenius ring.*

It follows from Theorem B that, for a division ring D, $\boldsymbol{C}(D;-1)$ and $\boldsymbol{H}(D;-1,-1)$ are quasi-Frobenius rings. Our motivation of this paper is to study these quasi-Frobenius rings.

Let R be a ring. In order to study the structure of $\boldsymbol{C}(R;-1)$ and $\boldsymbol{H}(R;-1,-1)$, we observe idempotents and nilpotents in these rings. For $\alpha = x+iy+jz+kw \in \boldsymbol{H}(R;-1,-1)$ $(x,y,z,w \in R)$, we write

$$\alpha^2 = A + iB + jC + kD$$

where $A, B, C, D \in R$. Then, by calculation, we see

$$\begin{aligned} A &= x^2 - y^2 - z^2 - w^2, \quad & B &= xy + yx + zw - wz, \\ C &= xz + zx + wy - yw, \quad & D &= wx + xw + yz - zy. \end{aligned}$$

Therefore,

$$\alpha^2 = 0 \iff$$

$$(\#) \begin{cases} x^2 - y^2 - z^2 - w^2 &= 0 \\ xy + yx + zw - wz &= 0 \\ xz + zx + wy - yw &= 0 \\ wx + xw + yz - zy &= 0. \end{cases}$$

Further,

$$\alpha^2 = \alpha \iff$$
$$(*) \begin{cases} x^2 - y^2 - z^2 - w^2 & = & x \\ xy + yx + zw - wz & = & y \\ xz + zx + wy - yw & = & z \\ wx + xw + yz - zy & = & w. \end{cases}$$

By $(*)$, we obtain

Fact 1. Let $R = F$ be a commutative field with $2 \neq 0$. Then

$$\alpha^2 = \alpha \notin \{0, 1\} \iff x = \frac{1}{2} \text{ and } \frac{1}{4} + y^2 + z^2 + w^2 = 0.$$

We first show the following theorem.

Theorem 1. *Let D be a division ring with $2 \neq 0$. Then*

(1) $J(\boldsymbol{H}(D; -1, -1)) = 0$ *and* $\boldsymbol{H}(D; -1, -1)$ *is a simple ring.*

(2) $|Pi(\boldsymbol{H}(D; -1, -1))| = 1, 2$ *or* 4.

(3) $|Pi(\boldsymbol{H}(D; -1 - 1))| = 1$ *iff* $\boldsymbol{H}(D; -1, -1)$ *is a division ring.*

(4) *Let* $|Pi(\boldsymbol{H}(D; -1 - 1))| = 2$. *Then, for any primitive idempotent* $e \in \boldsymbol{H}(D; -1, -1)$,

$$\boldsymbol{H}(D; -1, -1) \cong M_2(e\boldsymbol{H}(D; -1, -1)e).$$

(5) *Let* $|Pi(\boldsymbol{H}(D; -1 - 1))| = 4$. *Then, for any primitive idempotent* $e \in \boldsymbol{H}(D; -1, -1)$,

$$\boldsymbol{H}(D; -1, -1) \cong M_4(e\boldsymbol{H}(D; -1, -1)e).$$

In order to show $J(\boldsymbol{H}(D; -1, -1)) = 0$, we show the following lemma.

Lemma 2. *Let D be a division ring with $2 \neq 0$ and let $\alpha \in \boldsymbol{H}(D; -1, -1)$. If $\alpha^2 = (\alpha i)^2 = (\alpha j)^2 = (\alpha k)^2 = 0$, then $\alpha = 0$.*

Proof. Let $\alpha = x + iy + jz + kw \in \boldsymbol{H}(D; -1, -1)$ $(x, y, z, w \in D)$. By $\alpha^2 = 0$ and (#),

$$x^2 - y^2 - z^2 - w^2 = 0. \tag{2.1}$$

Since $\alpha i = -y + ix + jw - kz$ and $(\alpha i)^2 = 0$,

$$y^2 - x^2 - w^2 - z^2 = 0. \tag{2.2}$$

Similarly, by $(\alpha j)^2 = 0$ and $(\alpha k)^2 = 0$,

$$z^2 - w^2 - x^2 - y^2 = 0, \tag{2.3}$$

$$w^2 - z^2 - y^2 - x^2 = 0. \tag{2.4}$$

By (2.2)+(2.3)+(2.4)−(2.1), we have $x = 0$. Similarly, we obtain $y = z = w = 0$ and hence $\alpha = 0$, as required. □

Proof of Theorem 1 . (1) Suppose that $J(\boldsymbol{H}(D;-1,-1)) \neq 0$. Since $\boldsymbol{H}(D;-1,-1)$ is an artinian ring, there exists a non-zero simple right ideal $I \subseteq J(\boldsymbol{H}(D;-1,-1))$. Then, for any element $\alpha \in I$, $\alpha J(\boldsymbol{H}(D;-1,-1)) = 0$. By Lemma 2, we have $\alpha = 0$, a contradiction. Hence, $J(\boldsymbol{H}(D;-1,-1)) = 0$. Moreover, it is easily seen that $Z(\boldsymbol{H}(D;-1,-1)) \subseteq D$ and hence $0, 1$ are the only central idempotents in $\boldsymbol{H}(D;-1,-1)$. Therefore, $\boldsymbol{H}(D;-1,-1)$ is a simple ring.

(2) Let $|Pi(\boldsymbol{H}(D;-1,-1))| = n$ and

$$\boldsymbol{H}(D;-1,-1) = e_1\boldsymbol{H}(D;-1,-1) \oplus e_2\boldsymbol{H}(D;-1,-1) \oplus \cdots \oplus e_n\boldsymbol{H}(D;-1,-1)$$

where $\{e_1, e_2, \ldots, e_n\}$ is a complete set of orthogonal primitive idempotents. Since $\boldsymbol{H}(D;-1,-1)$ is a simple artinian ring by (1), we have

$$e_s\boldsymbol{H}(D;-1,-1)_{\boldsymbol{H}(D;-1,-1)} \cong e_t\boldsymbol{H}(D;-1,-1)_{\boldsymbol{H}(D;-1,-1)} \quad (1 \leq s, t \leq n).$$

In particular, $\dim_D e_s\boldsymbol{H}(D;-1,-1)_D = \dim_D e_1\boldsymbol{H}(D;-1,-1)_D$ $(2 \leq s \leq n)$. Hence,

$$n \cdot \dim_D e_1\boldsymbol{H}(D;-1,-1)_D = \dim_D \boldsymbol{H}(D;-1,-1)_D = 4,$$

which implies $n = 1, 2$ or 4.

(3) By (1) it is obvious.

(4) Let e be a primitive idempotent of $\boldsymbol{H}(D;-1,-1)$ and let $\{e_1 = e,\ e_2\}$ be a complete set of orthogonal primitive idempotents of $H(D;-1,-1)$. Then, as in the proof of (2), $e_2\boldsymbol{H}(D;-1,-1)_{\boldsymbol{H}(D;-1,-1)} \cong e\boldsymbol{H}(D;-1,-1)_{\boldsymbol{H}(D;-1,-1)}$. Hence,

$$\boldsymbol{H}(D;-1,-1) \cong M_2(e\boldsymbol{H}(D;-1,-1)e).$$

(5) is similarly shown as in (4). □

The following is an example of $|Pi(\boldsymbol{H}(D;-1,-1))| = 4$.

Example 3. Consider the Hamilton's quaternions

$$D := \boldsymbol{H}(\mathbb{R};-1,-1) = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R},$$

and

$$\boldsymbol{H}(D;-1,-1) = \boldsymbol{H}(\boldsymbol{H}(\mathbb{R};-1,-1);-1,-1) = D \oplus \mathrm{i}D \oplus \mathrm{j}D \oplus \mathrm{k}D.$$

Then, $|Pi(\boldsymbol{H}(D;-1,-1))| = 4$. In fact, put

$$g_1 = \frac{1}{4}(1 + \mathrm{i}i + \mathrm{j}j + \mathrm{k}k), \quad g_2 = \frac{1}{4}(1 + \mathrm{i}i - \mathrm{j}j - \mathrm{k}k),$$
$$g_3 = \frac{1}{4}(1 - \mathrm{i}i + \mathrm{j}j - \mathrm{k}k), \quad g_4 = \frac{1}{4}(1 - \mathrm{i}i - \mathrm{j}j + \mathrm{k}k).$$

Then, it can be easily checked that $\{g_1, g_2, g_3, g_4\}$ is a complete set of orthogonal primitive idempotents and hence $|Pi(\boldsymbol{H}(D;-1,-1))| = 4$.

For $|Pi(\boldsymbol{H}(D;-1,-1))| = 4$, we show the following theorem.

Theorem 4. *Let D be a division ring with $2 \neq 0$. The following conditions are equivalent:*

(i) $|Pi(\boldsymbol{H}(D;-1,-1))| = 4$.

(ii) *There exist $p, q \in D$ such that $p^2 = -1, q^2 = -1$ and $pq = -qp$.*

Proof. (i) $\Rightarrow$ (ii). Let $|Pi(\boldsymbol{H}(D;-1,-1))| = 4$ and $\{e_1, e_2, e_3, e_4\}$ be a complete set of orthogonal primitive idempotents of $\boldsymbol{H}(D;-1,-1)$. Noting that $\dim_D \boldsymbol{H}(D;-1,-1)_D = 4$, we have $e_\ell \boldsymbol{H}(D;-1,-1) = e_\ell D$ $(1 \leq \ell \leq 4)$. Hence, there exist $p, q \in D$ such that

$$e_1 i = e_1 p \quad \text{and} \quad e_1 j = e_1 q.$$

Then,

$$e_1 p^2 = (e_1 p)p = (e_1 i)p = e_1(ip) = e_1(pi) = e_1 ii = e_1(-1).$$

Since $p \in D$ and $e_1 \neq 0$, $p^2 = -1$. In a similar way, we obtain $q^2 = -1$. Moreover, $e_1(pq) = e_1 iq = e_1 qi = e_1 ji = -e_1 ij = -e_1 pj = -e_1 jp = -e_1 qp = e_1(-qp)$ and so $pq = -qp$.

(ii) $\Rightarrow$ (i). Assume that there exist $p, q \in D$ such that $p^2 = -1, q^2 = -1, pq = -qp$. Put

$$g_1 = \frac{1}{4}(1 + ip + jq + kpq), \quad g_2 = \frac{1}{4}(1 + ip - jq - kpq),$$
$$g_3 = \frac{1}{4}(1 - ip + jq - kpq), \quad g_4 = \frac{1}{4}(1 - ip - jq + kpq).$$

Then, we can check that $\{g_1, g_2, g_3, g_4\}$ is a complete set of orthogonal primitive idempotents of $\boldsymbol{H}(D;-1,-1)$. $\square$

By Theorems 1 and 4, we obtain the following result.

Corollary 5. *Let F be a commutative field with $2 \neq 0$. Then,*

$$|Pi(\boldsymbol{H}(F;-1,-1))| = 1 \text{ or } 2.$$

Therefore, $\boldsymbol{H}(F;-1,-1)$ is a division ring or $\boldsymbol{H}(F;-1,-1) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}$.

By Example 3 and Corollary 5, we can see different situations between the structures of $\boldsymbol{H}(F;-1,-1)$ and $\boldsymbol{H}(D;-1,-1)$.

Theorem 6. *Let F be a commutative field with $2 \neq 0$. Assume that $D = \boldsymbol{H}(F;-1,-1)$ is a division ring. Then*

$$\boldsymbol{H}(D;-1,-1) \cong \begin{pmatrix} F & F & F & F \\ F & F & F & F \\ F & F & F & F \\ F & F & F & F \end{pmatrix}.$$

In particular,

$$\boldsymbol{H}(\boldsymbol{H}(\mathbb{R};-1,-1);-1,-1) \cong \begin{pmatrix} \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \end{pmatrix}.$$

Proof. Let e be a primitive idempotent of $\boldsymbol{H}(D;-1,-1)$. Then, by Theorem 4, $|Pi(\boldsymbol{H}(D;-1,-1))| = 4$. Hence, by Theorem 1,

$$\boldsymbol{H}(D;-1,-1) \cong M_4(e\boldsymbol{H}(D;-1,-1)e).$$

Since $e\boldsymbol{H}(D;-1,-1)e \supseteq eDe \supseteq eFe = eF$, noting the dimensions of these rings over F, we see $e\boldsymbol{H}(D;-1,-1)e = eF \cong F$ as rings. Therefore the theorem follows. □

Theorem 7. *Let D be a division ring with $2 \neq 0$. Assume that there exists $x \in Z(D)$ such that $x^2 = -1$, that is, $x = \sqrt{-1} \in Z(D)$. Put $e = (1 - ix)2^{-1}$ and $f = (1 + ix)2^{-1}$. Then $\{e, f\}$ is a complete set of orthogonal primitive idempotents and*

$$\boldsymbol{H}(D;-1,-1) \cong \begin{pmatrix} D & D \\ D & D \end{pmatrix}.$$

Proof. Put $S = \{z \in D \mid z^2 = -1\}$. Then, for any $y \in S$, $(y + x)(y - x) = y^2 - x^2 = 0$. This implies that $y = \pm x \in Z(D)$ and $S = \{x, -x\} \subseteq Z(D)$. Hence, by Theorems 1 and 4, we can see that $|Pi(\boldsymbol{H}(D;-1,-1))| \leq 2$ and e and f are orthogonal primitive idempotents.

Let $h = p + iq + jr + ks \in \boldsymbol{H}(D;-1,-1)$ $(p, q, r, s \in D)$. Then, $ehe = e(p + xq) \in eD$. Hence, $e\boldsymbol{H}(D;-1,-1)e \subseteq eD$. Since $eD = eDe \subset e\boldsymbol{H}(D)e$, we have $e\boldsymbol{H}(D;-1,-1)e = eD = De \cong D$. Therefore, by Theorem 1,

$$\boldsymbol{H}(D;-1,-1) \cong \begin{pmatrix} e\boldsymbol{H}(D;-1,-1)e & e\boldsymbol{H}(D;-1,-1)e \\ e\boldsymbol{H}(D;-1,-1)e & e\boldsymbol{H}(D;-1,-1)e \end{pmatrix} \cong \begin{pmatrix} D & D \\ D & D \end{pmatrix}.$$

□

Corollary 8. *Let D be a division ring with $2 \neq 0$. If $\boldsymbol{C}(D;-1)$ is a division ring, then*

$$\boldsymbol{H}(\boldsymbol{C}(D;-1);-1,-1) \cong \begin{pmatrix} \boldsymbol{C}(D;-1) & \boldsymbol{C}(D;-1) \\ \boldsymbol{C}(D;-1) & \boldsymbol{C}(D;-1) \end{pmatrix}.$$

Comparing to theorem 1, we characterize $H(D;-1,-1)$ when D is a division ring with $2 = 0$.

Theorem 9. *Let D be a division ring with $2 = 0$. Then the following results hold:*

(1) $|Pi(\boldsymbol{H}(D;-1,-1))| = 1$.

(2) $\boldsymbol{H}(D;-1,-1)$ *is a local quasi-Frobenius ring such that*

$J(\boldsymbol{H}(D;-1,-1)) = (1+i)\boldsymbol{H}(D;-1,-1) + (1+j)\boldsymbol{H}(D;-1,-1)$ *and*
$S(\boldsymbol{H}(D;-1,-1)) = (1+i+j+k)\boldsymbol{H}(D;-1,-1)$.

Proof. (1) Let $e = x + iy + jz + kw$ $(x, y, z, w \in D)$ be an idempotent of $\boldsymbol{H}(D;-1,-1)$. By using $(*)$ stated in §2, we can see that $(x+y+z+w)^2 = x+y+z+w$ and hence $x+y+z+w = 0$ or $x+y+z+w = 1$. On the other hand, by the first and second equations of $(*)$, we have

$$(x+y)^2 + (z+w)^2 = x + y. \tag{2.5}$$

If $x+y+z+w = 0$, then $x+y = z+w$. Substituting this into (2.5), we obtain $x+y = 0$. This implies that $x = y$ and $z = w$. Hence, by $(*)$, $x = y = z = w = 0$ and $e = 0$. Let $x+y+z+w = 1$. Then, using the equation $z+w = 1+x+y$, in the same manner we can see that $x+y = 1$ and $z = w$. It follows from $(*)$ that $x = 1, y = z = w = 0$. Therefore $e = 1$.

(2) By (1) and Theorem B, $\boldsymbol{H}(D;-1,-1)$ is a local quasi-Frobenius ring. Further we see that $J(\boldsymbol{H}(D;-1,-1))=(1+i)\boldsymbol{H}(D;-1,-1)+(1+j)\boldsymbol{H}(D;-1,-1)$ and $S(\boldsymbol{H}(D;-1,-1)) = (1+i+j+k)\boldsymbol{H}(D;-1,-1)$. □

3 Structure of $\boldsymbol{C}(D;-1)$

By Theorem B, for a division ring D, $\boldsymbol{C}(D;-1)$ is a quasi-Frobenius ring and, in case $2 \neq 0$, it is a semisimple ring.

We show the following theorem.

Theorem 10. *For a division ring D, the following conditions are equivalent:*

(i) $x^2 \neq -1$ *for all* $x \in D$.

(ii) $\boldsymbol{C}(D;-1)$ *is a division ring.*

Proof. (i) $\Rightarrow$ (ii). Assume (i). Then, from the assumption, we have $2 \neq 0$. Suppose that $\boldsymbol{C}(D;-1)$ is not a division ring. Then there exist primitive idempotents $e, f \in \boldsymbol{C}(D;-1)$ such that $\boldsymbol{C}(D;-1) = e\boldsymbol{C}(D;-1) \oplus f\boldsymbol{C}(D;-1)$. Since $\boldsymbol{C}(D;-1)$ is a 2-dimensional D-space, $e\boldsymbol{C}(D;-1) = eD$, and hence there exists $x \in D$ such that $ei = ex$. Set $e = a + ib$ $(a, b \in D)$. Then, $ei = -b + ia$ and $ex = ax + ibx$. Hence $-b = ax$ and $a = bx$, and it follows $-1 = x^2$, a contradiction.

(ii) $\Rightarrow$ (i). Assume that $\boldsymbol{C}(D;-1)$ is a division ring. If $2 = 0$, then $(1+i)^2 = 0$, and hence $1+i = 0$, which is a contradiction. Hence $2 \neq 0$. Now, suppose that there exists $x \in D$ such that $x^2 = -1$. Then $e = (1 + ix)2^{-1}$ is an idempotent. Since $\boldsymbol{C}(D;-1)$ is a division ring, e must be 0 or 1, a contradiction. □

Remark 11. (1) The implication (i) $\Rightarrow$ (ii) is shown in [2] and Chapter 10 in [1]. Its proof we state below is complicated but above proof is a ring theoretic clear proof.

In fact, let $x = \alpha + i\beta$ $(\alpha, \beta \in D)$ be a non-zero element of $\boldsymbol{C}(D;-1)$. If $\beta = 0$, then $x^{-1} = \alpha^{-1}$. If $\beta \neq 0$, then

$$(\alpha + i\beta)(\beta^{-1}\alpha - i)\beta^{-1}((\alpha\beta^{-1})^2 + 1)^{-1} = 1,$$
$$((\beta^{-1}\alpha)^2 + 1)^{-1}(\beta^{-1}\alpha - i)\beta^{-1}(\alpha + i\beta) = 1.$$

Hence $x^{-1} = (\beta^{-1}\alpha - i)\beta^{-1}((\alpha\beta^{-1})^2 + 1)^{-1}$.

(2) Let $S := \{x \in D \mid x^2 = -1\}$. Then, as we saw in the proof of Theorem 7, if there exists $x \in Z(D)$ with $x^2 = -1$, then $S = \{x, -x\}$.

Theorem 12. *Let D be a division ring with $2 \neq 0$. Assume that there exists $x \in D$ such that $x^2 = -1$. Put $e = (1 + ix)2^{-1}$, $f = 1 - e = (1 - ix)2^{-1}$. Then $\boldsymbol{C}(D;-1) = e\boldsymbol{C}(D;-1) \oplus f\boldsymbol{C}(D;-1)$.*

(1) *If $x \in Z(D)$, then $\boldsymbol{C}(D;-1) = e\boldsymbol{C}(D;-1) \times f\boldsymbol{C}(D;-1)$ (ring direct sum).*

(2) *In the case $x \notin Z(D)$,*

$$\boldsymbol{C}(D;-1) \cong \begin{pmatrix} e\boldsymbol{C}(D;-1)e & e\boldsymbol{C}(D;-1)e \\ e\boldsymbol{C}(D;-1)e & e\boldsymbol{C}(D;-1)e \end{pmatrix}.$$

Proof. (1) If $x \in Z(D)$, then $e \in Z(\boldsymbol{C}(D;-1))$, whence we have the assertion.

(2) Assume that $x \notin Z(D)$ and take $d \in D$ such that $xd \neq dx$. Then by calculation, we have that $edf = (d + xdx + i(xd - dx))4^{-1}$, from which we see $edf \neq 0$. It follows $f\boldsymbol{C}(D;-1) \cong e\boldsymbol{C}(D;-1)$, and hence we obtain the ring isomorphism. □

Theorem 13. *Let D be a division ring with $2 = 0$. Then, $\boldsymbol{C}(D;-1)$ is a local quasi-Frobenius ring such that*

$$J(\boldsymbol{C}(D;-1)) = S(\boldsymbol{C}(D;-1)) = (1 + i)\boldsymbol{C}(D;-1).$$

Proof. By Theorem B, $\boldsymbol{C}(D;-1)$ is a quasi-Frobenius ring. Let $p = 1 + i$. Then, $p\boldsymbol{C}(D;-1)$ is nilpotent, because $p^2 = 0$ and $p \in Z(\boldsymbol{C}(D;-1))$. Noting that $\dim_D \boldsymbol{C}(D;-1)_D = 2$, we have $p\boldsymbol{C}(D;-1) = J(\boldsymbol{C}(D;-1)) = S(\boldsymbol{C}(D;-1))$ and $\boldsymbol{C}(D;-1)$ is a local ring. □

4 Structure of $\boldsymbol{C}(D;a)$

Let D be a division ring and $a \in Z(D) \setminus \{0\}$. First we extend Theorem 10 as follows.

Theorem 14. *Let D be a division ring. Then the following conditions are equivalent:*

(i) $x^2 \neq a$ *for all* $x \in D$.

(ii) $\boldsymbol{C}(D;a)$ *is a division ring.*

Proof. (i) $\Rightarrow$ (ii). Assume that $\boldsymbol{C}(D;a)$ is not a division ring. Then there exists $p \in \boldsymbol{C}(D;a)$ such that $0 \neq p\boldsymbol{C}(D;a) \subsetneqq \boldsymbol{C}(D;a)$. It follows from $\dim_D p\boldsymbol{C}(D;a)_D = 1$ that $p\boldsymbol{C}(D;a) = pD$. Hence there exists $x \in D$ such that $pi = px$. Then, $pa = pi^2 = (px)i = p(xi) = (pi)x = px^2$. Hence, we have $x^2 = a$.

(ii) $\Rightarrow$ (i). Assume that there exists $x \in D$ such that $x^2 = a$. Then $(x+i)(x-i) = x^2 - i^2 = 0$. Since $x + i \neq 0$ and $x - i \neq 0$, $\boldsymbol{C}(D;a)$ is not a division ring. □

By the same argument as in the proof of Theorem 12, we show the following theorem.

Theorem 15. *Let D be a division ring with $2 \neq 0$. Assume that there exists $x = \sqrt{a} \in D$. Put $e = (1 + ixa^{-1})2^{-1}$, $f = 1 - e = (1 - ixa^{-1})2^{-1}$. Then $\boldsymbol{C}(D;a) = e\boldsymbol{C}(D;a) \oplus f\boldsymbol{C}(D;a)$.*

(1) *If $x \in Z(D)$, then $\boldsymbol{C}(D;a) = e\boldsymbol{C}(D;a) \times f\boldsymbol{C}(D;a) \cong D \times D$ (ring direct sum).*

(2) *In the case $x \notin Z(D)$,*

$$\boldsymbol{C}(D;a) \cong \begin{pmatrix} e\boldsymbol{C}(D;a)e & e\boldsymbol{C}(D;a)e \\ e\boldsymbol{C}(D;a)e & e\boldsymbol{C}(D;a)e \end{pmatrix} = \begin{pmatrix} eDe & eDe \\ eDe & eDe \end{pmatrix}.$$

Theorem 16. *Let D be a division ring with $2 = 0$ and there exists $x = \sqrt{a} \in D$.*

(1) *If $x \in Z(D)$, then $\boldsymbol{C}(D;a)$ is a local quasi-Frobenius ring such that*

$$J(\boldsymbol{C}(D;a)) = S(\boldsymbol{C}(D;a)) = (x+i)\boldsymbol{C}(D;a).$$

(2) *Assume that* $x \notin Z(D)$ *and take* $d \in D$ *with* $xd \neq dx$. *Then,*

$$\boldsymbol{C}(D;a) \cong \begin{pmatrix} e\boldsymbol{C}(D;a)e & e\boldsymbol{C}(D;a)e \\ e\boldsymbol{C}(D;a)e & e\boldsymbol{C}(D;a)e \end{pmatrix} = \begin{pmatrix} eDe & eDe \\ eDe & eDe \end{pmatrix}$$

where $e = (x+i)d(xd+dx)^{-1}$.

Proof. (1) Let $x \in Z(D)$ with $x^2 = a$ and put $p = x + i$. Then, $p\boldsymbol{C}(D;a)$ is nilpotent, because $p^2 = 0$ and $p \in Z(\boldsymbol{C}(D;a))$. Noting that the dimension, we have $p\boldsymbol{C}(D;a) = J(\boldsymbol{C}(D;a))$ and it is the only non-zero proper right (left) ideal of $\boldsymbol{C}(D;a)$. Hence $\boldsymbol{C}(D;a)$ is a local ring and, by noting its right socle and left socle are simple, we see that $\boldsymbol{C}(D;a)$ is a quasi-Frobenius ring (see Nicholson-Yousif [8]).

(2) Let $x \in D \setminus Z(D)$ with $x^2 = a$. Moreover, let $d \in D$ such that $xd \neq dx$ and $t = xd + dx$. Then, $xt = ad + xdx = da + xdx = tx$ and $dt^{-1}(x+i) = dxt^{-1} + dt^{-1}i = (xd+t)t^{-1} + dt^{-1}i = (x+i)dt^{-1} + 1$. Putting $e = (x+i)dt^{-1}$, we see $e^2 = (x+i)^2(dt^{-1})^2 + (x+i)dt^{-1} = e$. Since it is easily seen that $ex(1-e) = x + i \neq 0$, $eD = e\boldsymbol{C}(D;a) \cong (1-e)\boldsymbol{C}(D;a)$. Thus we have the assertion. □

5 Structure of $\boldsymbol{H}(D;a,b)$

Let R be a ring and $a, b \in Z(R) \setminus \{0\}$. In order to study the structure of $\boldsymbol{H}(R;a,b)$, we observe idempotents and nilpotents in these rings as in Section 2.

For $\alpha = x + iy + jz + kw \in \boldsymbol{H}(R;a,b)$ $(x,y,z,w \in R)$, we write

$$\alpha^2 = A + iB + jC + kD$$

where $A, B, C, D \in R$. Then, by calculation, we see

$$\begin{aligned} A &= x^2 + ay^2 + bz^2 - abw^2, \quad & B &= xy + yx - bzw + bwz, \\ C &= xz + zx + ayw - awy, \quad & D &= wx + xw + yz - zy. \end{aligned}$$

Therefore,

$$\alpha^2 = 0 \iff$$
$$(\#2) \begin{cases} x^2 + ay^2 + bz^2 - abw^2 & = & 0 \\ xy + yx - bzw + bwz & = & 0 \\ xz + zx + ayw - awy & = & 0 \\ wx + xw + yz - zy & = & 0. \end{cases}$$

Further,

$$\alpha^2 = \alpha \iff$$
$$(*2)\begin{cases} x^2 + ay^2 + bz^2 - abw^2 & = & x \\ xy + yx - bzw + bwz & = & y \\ xz + zx + ayw - awy & = & z \\ wx + xw + yz - zy & = & w. \end{cases}$$

By $(*2)$ above, we obtain:

Fact 2. Let $R = F$ be a commutative field with $2 \neq 0$. Then

$$\alpha^2 = \alpha \notin \{0, 1\} \iff x = \frac{1}{2} \text{ and } \frac{1}{4} - ay^2 - bz^2 + abw^2 = 0.$$

Here we state some results on a generalized quaternion ring $\boldsymbol{H}(D; a, b)$, where D is a division ring with $2 \neq 0$. By $(*2)$ above and using the same arguments as in the previous sections, we can show the following theorems which correspond to Theorem 1, Corollary 5 and Theorem 4.

Theorem 17. *Let D be a division ring with $2 \neq 0$.*

(1) *$J(\boldsymbol{H}(D; a, b)) = 0$ and $\boldsymbol{H}(D; a, b)$ is a simple ring.*

(2) *$|Pi(\boldsymbol{H}(D; a, b))| = 1$ or 2 or 4.*

(3) *$|Pi(\boldsymbol{H}(D; a, b))| = 1$ iff $\boldsymbol{H}(D; a, b)$ is a division ring.*

(4) *Let $|Pi(\boldsymbol{H}(D; a, b))| = 2$. For any primitive idempotent $e \in \boldsymbol{H}(D; a, b)$,*

$$\boldsymbol{H}(D; a, b) \cong \begin{pmatrix} e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e \\ e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e \end{pmatrix}.$$

(5) *Let $|Pi(\boldsymbol{H}(D; a, b))| = 4$. For any primitive idempotent $e \in \boldsymbol{H}(D; a, b)$,*

$$\boldsymbol{H}(D; a, b) \cong \begin{pmatrix} e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e \\ e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e \\ e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e \\ e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e & e\boldsymbol{H}(D; a, b)e \end{pmatrix}.$$

(6) *If $D = F$ is a commutative field, then $\boldsymbol{H}(F; a, b)$ is a division ring or $\boldsymbol{H}(F; a, b) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}$.*

(7) *For a commutative field F, $|Pi(\boldsymbol{H}(F; a, b))| = 4$ does not occur.*

The following theorem is one of main results of this paper.

Theorem 18. *Let D be a division ring with $2 \neq 0$. The following conditions are equivalent:*

(i) $|Pi(\boldsymbol{H}(D; a, b))| = 4$.

(ii) *There exist $p, q \in D$ such that $p^2 = a, q^2 = b$ and $pq = -qp$.*

In these case, the following $\{g_1, g_2, g_3, g_4\}$ is a complete set of orthogonal primitive idempotents:

$$\begin{aligned}
g_1 &= \frac{1}{4}(1 + ipa^{-1} + jqb^{-1} + kpq(ab)^{-1}),\\
g_2 &= \frac{1}{4}(1 + ipa^{-1} - jqb^{-1} - kpq(ab)^{-1}),\\
g_3 &= \frac{1}{4}(1 - ipa^{-1} + jqb^{-1} - kpq(ab)^{-1}),\\
g_4 &= \frac{1}{4}(1 - ipa^{-1} - jqb^{-1} + kpq(ab)^{-1}).
\end{aligned}$$

Using this theorem, we can obtain examples $\boldsymbol{H}(D; a, b)$ with $|Pi(\boldsymbol{H}(D; a, b))| = 4$.

Example 19 *(cf. Theorem 6).* Consider $D = \boldsymbol{H}(\mathbb{R}; a, b)$ where $a, b < 0$. Since the solution of the equation $X^2 - aY^2 - bZ^2 = 0$ is only $(0, 0, 0)$, we can see from Theorem 17 and Theorem 21 below that D is a division ring. For this D, by the theorem above, we see $|Pi(\boldsymbol{H}(D; a, b))| = 4$ and

$$\boldsymbol{H}(D; a, b) \cong \begin{pmatrix} \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \end{pmatrix}.$$

Corresponding to Theorem 16 we obtain the following result.

Theorem 20. *Let D be a division ring with $2 = 0$ and put $H = \boldsymbol{H}(D; a, b)$. If there exist $\sqrt{a} \in Z(D)$ and $\sqrt{b} \in Z(D)$, then H is a local quasi-Frobenius ring such that*

$$\begin{aligned}
J(H) &= (\sqrt{a} + i)H + (\sqrt{b} + j)H\\
&= (\sqrt{a} + i)D \oplus (\sqrt{b} + j)D \oplus (\sqrt{a}\sqrt{b} + k)D \text{ and}\\
S({}_H H) &= S(H_H) = (\sqrt{a}\sqrt{b} + i\sqrt{b} + j\sqrt{a} + k)H\\
&= (\sqrt{a}\sqrt{b} + i\sqrt{b} + j\sqrt{a} + k)D.
\end{aligned}$$

Proof. Let $I = (\sqrt{a} + i)H + (\sqrt{b} + j)H$. Then, we can verify that

$$I = (\sqrt{a} + i)D \oplus (\sqrt{b} + j)D \oplus (\sqrt{a}\sqrt{b} + k)D,$$
$$I^2 = (\sqrt{a}\sqrt{b} + i\sqrt{b} + j\sqrt{a} + k)D = (\sqrt{a}\sqrt{b} + i\sqrt{b} + j\sqrt{a} + k)H$$

and $I^3 = 0$ by direct calculation. Since $\dim(H/I)_D = 1$, $I = J(H)$ and H is a local ring. Moreover, for any $\alpha = x + iy + jz + kw \in S({}_H H)$ $(x, y, z, w \in D)$,

$$0 = (\sqrt{a} + i)\alpha = (\sqrt{a}x + ay) + i(\sqrt{a}y + x) + j(\sqrt{a}z + aw) + k(\sqrt{a}w + z)$$

and

$$0 = (\sqrt{b} + j)\alpha = (\sqrt{b}x + bz) + i(\sqrt{b}y + bw) + j(\sqrt{b}z + x) + k(\sqrt{b}w + y).$$

It follows that $x = \sqrt{a}y, z = \sqrt{a}w$ and $y = \sqrt{b}w$. Substituting these equations into α, we have $\alpha = (\sqrt{a}\sqrt{b} + i\sqrt{b} + j\sqrt{a} + k)w \in I^2$. Hence $S({}_H H) = I^2$. Similarly, we obtain $S(H_H) = I^2$. Since $S({}_H H)$ and $S(H_H)$ are simple and H is a local artinian ring, H is a local quasi-Frobenius ring. □

We shall comment consistency between classical theory on $\boldsymbol{H}(F; a, b)$ and our theory on $Pi(\boldsymbol{H}(F; a, b))$, where F is a commutative field with $2 \neq 0$.

The following is a classical theorem on $\boldsymbol{H}(F; a, b)$ with $2 \neq 0$.

Theorem 21. *Let F is a commutative field with $2 \neq 0$. The following conditions are equivalent:*

(i) $\boldsymbol{H}(F; a, b) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}$.

(ii) *The equation $X^2 - aY^2 - bZ^2 + abW^2 = 0$ has a non-trivial solution in F.*

(iii) *The equation $X^2 - aY^2 - bZ^2 = 0$ has a non-trivial solution in F.*

However, in general, the following condition is not equivalent to these conditions.

(iv) $X^2 - aY^2 = 0$, *or* $X^2 - bZ^2 = 0$, *or* $X^2 + abW^2 = 0$ *has a solution in F.*

Indeed, for example, for $\boldsymbol{H}(\mathbb{Q}(\sqrt{-3}); -1, -1)$, we can easily see that (ii) holds but (iv) does not hold.

On the other hand, from our theory, we can show the following result, from which the classical theorem above follows.

Theorem 22. *Let F be a commutative field with $2 \neq 0$. Then, the following conditions are equivalent:*

(i) $\boldsymbol{H}(F;a,b) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}$.

(ii) *The equation* $\frac{1}{4} - aY^2 - bZ^2 + abW^2 = 0$ *has a solution in* F*, equivalently, the equation* $1 - aY^2 - bZ^2 + abW^2 = 0$ *has a solution in* F.

(iii) *There exists an idempotent* e *of the form* $e = \frac{1}{2} + iy + jz + kw \in \boldsymbol{H}(F;a,b)$.

(iv) *The equation* $\frac{1}{4} - aY^2 - bZ^2 = 0$ *has a solution in* F*, or the equation* $\frac{1}{4} + abW^2 = 0$ *has a solution in* F.

(v) *The equation* $\frac{1}{4} - bZ^2 + abW^2 = 0$ *has a solution in* F*, or the equation* $\frac{1}{4} - aY^2 = 0$ *has a solution in* F.

(vi) *The equation* $\frac{1}{4} - aY^2 + abW^2 = 0$ *has a solution in* F*, or the equation* $\frac{1}{4} - bZ^2 = 0$ *has a solution in* F.

(vii) *There exists an idempotent* e *of the form* $e = \frac{1}{2} + iy + jz \in \boldsymbol{H}(F;a,b)$*, or an idempotent* e *of the form* $e = \frac{1}{2} + kw \in \boldsymbol{H}(F;a,b)$.

(viii) *There exists an idempotent* e *of the form* $e = \frac{1}{2} + jz + kw \in \boldsymbol{H}(F;a,b)$*, or an idempotent* e *of the form* $e = \frac{1}{2} + iy \in \boldsymbol{H}(F;a,b)$.

(ix) *There exists an idempotent* e *of the form* $e = \frac{1}{2} + iy + kw \in \boldsymbol{H}(F;a,b)$*, or an idempotent* e *of the form* $e = \frac{1}{2} + jz \in \boldsymbol{H}(F;a,b)$.

(x) *At least one of the equations* $\frac{1}{4} - aY^2 - bZ^2 = 0$*,* $\frac{1}{4} - bZ^2 + abW^2 = 0$*, or* $\frac{1}{4} - aY^2 + abW^2 = 0$ *has a solution in* F.

(xi) *There exists an idempotent* e *of the form* $e = \frac{1}{2} + iy + jz$*,* $e = \frac{1}{2} + jz + kw$*, or* $e = \frac{1}{2} + iy + kw$ *in* $\boldsymbol{H}(F;a,b)$.

Proof. (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (ii) and (iv) $\Leftrightarrow$ (vii), (v) $\Leftrightarrow$ (viii), (vi) $\Leftrightarrow$ (ix) and (x) $\Leftrightarrow$ (xi) follow from Theorem 17 and Fact 2. Moreover, (iv) $\Rightarrow$ (ii) and (vii) $\Rightarrow$ (xi) $\Rightarrow$ (iii) are obvious.

(iv) $\Rightarrow$ (v). Assume the equation $\frac{1}{4} - aY^2 - bZ^2 = 0$ has a solution in F, say (y, z). If $z \neq 0$, then $(Z, W) = ((4bz)^{-1}, y(2bz)^{-1})$ is a solution of the equation $\frac{1}{4} - bZ^2 + abW^2 = 0$. Otherwise $Y = y$ is a solution of $\frac{1}{4} - aY^2 = 0$. On the other hand, in case the equation $\frac{1}{4} + abW^2 = 0$ has a solution in F, clearly so does $\frac{1}{4} - bZ^2 + abW^2 = 0$.

(v) $\Rightarrow$ (vi) and (vi) $\Rightarrow$ (iv) are similarly shown as in the above proof.

(ii) $\Rightarrow$ (iv). Let (y, z, w) be a solution of the equation $\frac{1}{4} - aY^2 - bZ^2 + abW^2 = 0$. In case $z \neq 2ayw$, put $y' = (2yz - w)(2z - 4ayw)^{-1}, z' = (z^2 - aw^2)(z -$

$2ayw)^{-1}$. Then,

$$\begin{aligned}
&\frac{1}{4} - ay'^2 - bz'^2 \\
&= (z^2 - aw^2)(\frac{1}{4} - ay^2 - bz^2 + abw^2)(z - 2ayw)^{-2} \\
&= 0.
\end{aligned}$$

Hence, (y', z') is a solution of the equation $\frac{1}{4} - aY^2 - bZ^2 = 0$. Let $z = 2ayw$. Then, since

$$4\left(\frac{1}{4} - ay^2\right)\left(\frac{1}{4} + abw^2\right) = \frac{1}{4} - ay^2 - bz^2 + abw^2 = 0,$$

we have $\frac{1}{4} - ay^2 = 0$ or $\frac{1}{4} + abw^2 = 0$. These equations imply that $(y, 0)$ and w are solutions of the first equation and the second equation in (iv), respectively. $\square$

Finally we give the following supplementary results. Let D be a division ring with $2 \neq 0$. For $w \in D$, we can see the following.

(1) $(1 + iw)2^{-1}$ in $\boldsymbol{H}(D; a, b)$ is an idempotent iff $w^2 = a^{-1}$.

(2) $(1 + jw)2^{-1}$ in $\boldsymbol{H}(D; a, b)$ is an idempotent iff $w^2 = b^{-1}$.

(3) $(1 + kw)2^{-1}$ in $\boldsymbol{H}(D; a, b)$ is an idempotent iff $w^2 = (-ab)^{-1}$.

In particular, if $w \in Z(D)$, then we can show, by a similar argument as in the proof of Theorem 7 using Theorems 17 and 18, that each idempotent above is a primitive idempotent. Therefore we obtain the following facts, etc.

$$\boldsymbol{H}(D; a, -a) \cong \boldsymbol{H}(D; -a, a) \cong \begin{pmatrix} D & D \\ D & D \end{pmatrix},$$
$$\boldsymbol{H}(D; a, -a^3) \cong \boldsymbol{H}(D; a^3, -a) \cong \begin{pmatrix} D & D \\ D & D \end{pmatrix}.$$

Moreover, we obtain the following example.

Example 23. Let D be a division ring with $2 \neq 0$. If $\sqrt{a} \in Z(D)$, $\sqrt{b} \in Z(D)$, or $\sqrt{-ab} \in Z(D)$, then

$$\boldsymbol{H}(D; a, b) \cong \begin{pmatrix} D & D \\ D & D \end{pmatrix}.$$

We give several mappings which define this isomorphism.

Let $x, y, z, w \in D$. Then, each mapping in (1) - (3) below defines an isomorphism from $\boldsymbol{H}(D; a, b)$ to $\begin{pmatrix} D & D \\ D & D \end{pmatrix}$.

(1) If $\sqrt{a} \in Z(D)$, then $(1 + i/\sqrt{a})2^{-1}$ is a primitive idempotent and the mapping

$$x + iy + jz + kw \mapsto \begin{pmatrix} x + y\sqrt{a} & (z + w\sqrt{a})\sqrt{a}b \\ (z - w\sqrt{a})/\sqrt{a} & x - y\sqrt{a} \end{pmatrix}$$

gives an isomorphism.

(2) If $\sqrt{b} \in Z(D)$, then $(1 + i/\sqrt{b})2^{-1}$ is a primitive idempotent and the mapping

$$x + iy + jz + kw \mapsto \begin{pmatrix} x + z\sqrt{b} & (y - w\sqrt{b})a\sqrt{b} \\ (y + w\sqrt{b})/\sqrt{b} & x - z\sqrt{b} \end{pmatrix}$$

gives an isomorphism.

(3) If $\sqrt{-ab} \in Z(D)$, then $(1 + i/\sqrt{-ab})2^{-1}$ is a primitive idempotent and the mapping

$$x + iy + jz + kw \mapsto \begin{pmatrix} x + w\sqrt{-ab} & y\sqrt{-ab} + zb \\ (-y\sqrt{-ab} + zb)/b & x - w\sqrt{-ab} \end{pmatrix}$$

gives an isomorphism.

We give a sketch of the proof of (1). Put $H = \boldsymbol{H}(D; a, b)$, $e = (1 + i/\sqrt{a})2^{-1}$ and $f = 1 - e = (1 - i/\sqrt{a})2^{-1}$. Then, $H = \begin{pmatrix} eHe & eHf \\ fHe & fHf \end{pmatrix}$, and we see $eHe = eD \cong D$. Hence e and f are primitive idempotents. For $P = x + iy + jz + kw$ $(x, y, z, w \in D)$, $ePe = e(x + \sqrt{a}y)$, $ePf = \alpha(\sqrt{a}bz + abw)$, $fPe = \beta(z/\sqrt{a} - w)$ and $fPf = f(x - \sqrt{a}y)$ where $\alpha = (j\sqrt{a} + k)(2ab)^{-1}, \beta = (j\sqrt{a} - k)2^{-1}$. Therefore,

$$P = \begin{pmatrix} e(x + \sqrt{a}y) & \alpha(\sqrt{a}bz + abw) \\ \beta(z/\sqrt{a} - w) & f(x - \sqrt{a}y) \end{pmatrix}.$$

Thus we get the isomorphism (1) above.

Acknowledgements We would like to thank Dr. G. Lee for his useful comments for the preliminary version of this paper. This work was supported in part by JSPS KAKENHI Grant Number 26400047.

References

[1] Y. Baba and K. Oshiro, "Classical Artinian Rings and Related Topics", World Sci. Publ., 2009

[2] I. Kikumasa, K. Oshiro and H. Yoshimura, *A construction of local QF-rings with radical cubed zero,* Proceedings of the 39th Symposium on Ring Theory and Representation Theory, Hiroshima Japan, 36–44, 2007

[3] I. Kikumasa, K. Koike and K. Oshiro, *Complex rings, Quaternion rings and Octonion rings,* Proceedings of the 50th Symposium on Ring Theory and Representation Theory, Yamanashi Japan, 74–83, 2018

[4] G. Lee and K. Oshiro, *Quaternion rings and octonion rings,* Frontiers of Mathematics in China, **12**(1), 143–155, 2017

[5] T. Y. Lam, “Lectures on Modules and Rings”, Graduate Texts in Mathematics, Springer-Verlag, New York, Heidelberg, Berlin, 1991

[6] T. Y. Lam, “The Algebraic Theory of Quadratic Forms”, Reading: Addison Wesley-Benjamin, 1973

[7] W. K. Nicholson, “Introduction to Abstract Algebra”, The Prindle, Weber and Schmidt Series in Mathematics PWS-KENT Publishing Company, Boston, 1993

[8] W. K. Nicholson and M. Y. Yousif, “Quasi-Frobenius rings”, Cambridge Tracts in Mathematics. Cambridge University Press. Cambridge, 2003

[9] R. S. Pierce, “Associative Algebras”, Springer-Verlag, New York, Heidelberg, Berlin, 1982