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γ-SMALL SUBMODULES AND γ-LIFTING MODULES

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Abstract

In this paper we introduce a new generalization of small submodules, namely γ-small submodules. We call a submodule K of a module M, γ-small provided $M = K + L$ with M/L noncosingular, implies $M = L$. Applying this concept, we define a generalization of lifting modules entitled γ-lifting modules and investigate their some general properties. It is proved that any supplement submodule of a γ-lifting module is γ-lifting.

1 Introduction

Let M be a module and L a submodule of M (we denote it by $L \leq M$). Then L is said to be *small* in M (denoted by $L \ll M$) in case $L + T \neq M$ for every proper submodule T of M. The module M is called *lifting*, in case for every submodule N of M there is a direct summand D of M contained in N such that $N/D \ll M/D$. We say that a submodule N of M is *supplement* in M, if there is a submodule K of M such that $M = N + K$ and $N \cap K \ll N$. Also M is called *supplemented* provided every submodule of M has a supplement in M. As a generalization of supplemented modules, a module M is said to be *amply supplemented* if $M = N + K$ implies N has a supplement L which is contained in K.

A module M is called *small* if there exist modules $L \leq K$ such that $M \cong L \ll K$. For a ring R and a right R-module M let $\overline{Z}(M) = Rej_M(\mathcal{S}) =$

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$\bigcap\{Kerf \mid f: M \to U, U \in \mathcal{S}\} = \bigcap\{K \subseteq M \mid M/K \in \mathcal{S}\}$ where $\mathcal{S}$ denotes the class of all small right R-modules. If $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$), then M is called a *cosingular* (*noncosingular*) module (see [3]). Note that by $\overline{Z}^2(M)$ we mean $\overline{Z}(\overline{Z}(M))$.

In last decades, small submodules and relevant concepts were widely studied and investigated. Many researchers tried to introduce and consider some notions in module theory closely related to smallness. Undoubted, one of the most famous concept in the theory of rings and modules is lifting modules. Maybe firstly, this concept introduced in the 1970s. After that we have a large number of works which their main subjects were lifting modules and their various generalizations (for example, [1]).

Zhou in [5] introduced a generalization of small submodules namely δ-small submodules via the concept of singular modules. In fact, he called a submodule N of a module M a *δ-small* submodule if $M \neq N + K$ for every proper submodule K of M with M/K singular. General properties of δ-small submodules and a nice characterization of them are also provided in [5]. He defined $\delta(M)$ for a module M to be $Rej_M(\mathcal{U})$ where $\mathcal{U}$ stands for the class of all simple singular right R-modules. This is a motivation for our study here to introduce a new generalization of small submodules. In fact, we consider the class of all simple noncosingular (injective) right R-modules in the definition of $\delta(M)$ and as a consequence in the definition of δ-small submodules we should change singular modules to noncosingular modules. By the way, we call a submodule N of a module M, γ-small provided $M \neq N + K$ for all proper submodules K of M with M/K noncosingular. We define $\gamma(M)$ to be sum of all δ-small submodules of M. We also show that $\gamma(M)$ is equal to $Rej_M(\mathcal{SN})$ where $\mathcal{SN}$ stands for the class of all simple noncosingular (injective) right R-modules. We try to study some natural and general properties of γ-small submodules. γ-coclosed submodules are introduced and their some natural properties are studied. As an application, we define γ-lifting modules. We say a module M is γ-lifting if for every submodule N of M there is a direct summand D of M contained in N such that N/D is γ-small in M/D. It is shown that a supplement submodule of a γ-lifting module is γ-lifting.

In what follows, $J(R)$ denotes the Jacobson radical of a ring R and $Rad(M)$ stands for the radical of a module M. For any unexplained terminologies we refer to [2].

2 γ-small submodules and γ-coclosed submodules

We start this section by providing the definition of a new generalization of small submodules. If in the definition of small submodules, we restrict submodules of the module the those, whose natural factor modules are noncosingular, we produce the following.

Definition 2.1. Let N be a submodule of M. Then we say that N is γ-small in M,(denoted by $N \ll_\gamma M$) if $M = N + X$ with M/X noncosingular implies $M = X$. In other words, $M \neq N + X$ for every proper submodule X of M with M/X noncosingular.

It is clear that every small submodule of a module is γ-small in that module.

We list some properties of γ-small submodules that are similar to those for small submodules.

Proposition 2.2. *Let M be an R-module. Then the following statements hold.*

(1) *Let $A \leq B \leq M$. Then $B \ll_\gamma M$ if and only if $A \ll_\gamma M$ and $\frac{B}{A} \ll_\gamma \frac{M}{A}$.*

(2) *Let A, B be submodules of M with $A \leq B$. If $A \ll_\gamma B$, then $A \ll_\gamma M$.*

(3) *Let $f : M \to M'$ be an epimorphism such that $A \ll_\gamma M$, then $f(A) \ll_\gamma M'$.*

(4) *Let $M = M_1 \oplus M_2$ be an R-module and let $A_1 \leq M_1$ and $A_2 \leq M_2$. Then $A_1 \oplus A_2 \ll_\gamma M_1 \oplus M_2$ if and only if $A_1 \ll_\gamma M_1$ and $A_2 \ll_\gamma M_2$.*

(5) *Let M be an R-module and $A \leq B$. If B is a supplement submodule in M and $A \ll_\gamma M$, then $A \ll_\gamma B$.*

Proof. (1) ($\Rightarrow$) Suppose that $B \ll_\gamma M$ and let U be a submodule of M such that $M = A + U$ with M/U noncosingular. Since $A \leq B$, then $M = B + U$. Being B a γ-small submodule of M implies $M = U$. Thus $A \ll_\gamma M$. Now assume that $M/A = B/A + L/A$ for some submodule L of M and $\frac{M/A}{L/A} \cong M/L$ is noncosingular. Then $M = B + L$ combining with $B \ll_\gamma M$ yields that $M = L$.

($\Leftarrow$) Suppose that $A \ll_\gamma M$ and $B/A \ll_\gamma M/A$. To prove $B \ll_\gamma M$ suppose $M = B + U$ with M/U noncosingular. So $M/A = B/A + (U + A)/A$. Note that $\frac{M/A}{(U+A)/A} \cong M/(U + A)$ is noncosingular. Since $B/A \ll_\gamma M/A$, then $M/A = (U + A)/A$ which implies that $M = U + A$. As $A \ll_\gamma M$ and M/U is noncosingular we conclude that $M = U$. It follows that $B \ll_\gamma M$.

(2) Suppose that $A \ll_\gamma B$. Let $M = A+U$ such that M/U is noncosingular. Since $B = B \cap M = B \cap (A + U) = A + (B \cap U)$ (by modular law), we have $B/(B \cap U) \cong (B + U)/U = M/U$ which implies $B/(B \cap U)$ is noncosingular. By $A \ll_\gamma B$ we conclude that $B = B \cap U$. Hence $M = U$.

(3) Let $A \ll_\gamma M$ and $f(A) + Y = M'$ for a submodule Y of M' such that M'/Y is noncosingular. It is easy to check that $A + f^{-1}(Y) = M$. Being $M/f^{-1}(Y)$ a homomorphic image of M'/Y implies $M/f^{-1}(Y)$ is noncosingular. Hence $M = f^{-1}(Y)$. It is easy to verify that $M' = Y$.

(4) ($\Rightarrow$) Suppose that $A_1 \oplus A_2 \ll_\gamma M_1 \oplus M_2$. Let $p : M_1 \oplus M_2 \to M_1$ be the projection on M_1. Since $A_1 \oplus A_2 \ll_\gamma M_1 \oplus M_2$, then $p(A_1 \oplus A_2) \ll_\gamma p(M_1 \oplus M_2)$ by (3). It follows that $A_1 \ll_\gamma M_1$. Similarly $A_2 \ll_\gamma M_2$.

($\Leftarrow$) Suppose that $A_1 \ll_\gamma M_1$ and $A_2 \ll_\gamma M_2$. Let $A_1+A_2+X = M_1+M_2$ with $\frac{(M_1+M_2)}{X}$ noncosingular. So $(M_1+M_2)/(A_2+X)$ as a homomorphic image of $(M_1+M_2)/X$, is noncosingular. Since $A_1 \ll_\gamma M_1+M_2$ by (2), we conclude that $A_2 + X = M_1 + M_2$. Now $A_2 \ll_\gamma M_1 + M_2$ implies $X = M_1 + M_2$ as required.

(5) Let $A \ll_\gamma M$ and B be a supplement submodule of B' in M. Then $M = B + B'$ and $B \cap B' \ll B$. To show that $A \ll_\gamma B$, let $B = A + U$ such that B/U is noncosingular. Then $M = B + B' = A + U + B'$. Since $M/(U+B') = (A+U+B')/(U+B') \cong A/(A\cap(U+B'))$ and $A/(A\cap(U+B'))$ is a homomorphic image of $A/(A\cap U) \cong B/U$, then it will be noncosingular. Hence $M = U + B'$ as $A \ll_\gamma M$. Now being $B \cap B'$ a small submodule of B implies $B = B \cap M = B \cap (U + B') = U + (B \cap B') = U$. It follows that $A \ll_\gamma B$. □

The following provides a characterization of a module M such that every submodule of M is γ-small in M.

Proposition 2.3. *Let M be a module. Consider the following:*

(1) *$M \ll_\gamma M$;*

(2) *Each submodule of M is γ-small in M;*

(3) *Non of nonzero homomorphic images of M is noncosingular;*

(4) *$\overline{Z}(M) \ll M$.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4). *They are equivalent in case, M is amply supplemented.*

Proof. (1) $\Rightarrow$ (2) It follows from Proposition 2.2(1).

(2) $\Rightarrow$ (3) Suppose that every submodule of M is γ-small in M. Consider a submodule X of M such that M/X is noncosingular. Since $M = M + X$ and $M \ll_\gamma M$, then $M = X$.

(3) $\Rightarrow$ (1) Let X be a submodule of M such that M/X is noncosingular. By assumption $X = M$ which shows that $M \ll_\gamma M$.

(3) $\Rightarrow$ (4) Let X be a proper submodule of M. Then $\overline{Z}(M/X) \neq M/X$. It is easy to check that $\overline{Z}(M) + X \neq M$ which implies that $\overline{Z}(M) \ll M$.

(4) $\Rightarrow$ (3) Let M be amply supplemented and $\overline{Z}(M) \ll M$. Suppose that M/X be a noncosingular homomorphic image of M. Then $M/X = \overline{Z}(M/X) = \overline{Z}^2(M/X) = \frac{\overline{Z}^2(M)+X}{X}$. Since $\overline{Z}(M)$ is a cosingular module, then $\overline{Z}^2(M) = 0$. Therefore, $M/X = 0$. □

It is clear that every small submodule of a module is γ-small. We provide some examples to indicate that the converse may not hold.

Example 2.4. (1) *Let $M = \mathbb{Z}$ as a module over itself. Since every homomorphic image of M is cosingular, then every submodule of M is γ-small in M by Proposition 2.3. Note that non of nonzero submodules of M is small in M.*

(2) *Let M be a semisimple cosingular module. Since every nonzero homomorphic image of M is cosingular, then every submodule of M is γ-small in M by Proposition 2.3 while the only small submodule of M is the zero submodule.*

The following presents some conditions which under two concepts small and γ-small coincide.

Proposition 2.5. *Let M be a module and $N \leq M$. Then in each of the following cases $N \ll M$ if and only if $N \ll_\gamma M$:*
(1) *N is noncosingular.*
(2) *M is noncosingular.*
(3) *$N/D \ll M/D$ where D is a noncosingular direct summand of M.*

Proof. (1) Let $N \ll_\gamma M$ and N be noncosingular. Let also $N + X = M$. Then we have $\frac{N}{N\cap X} \cong \frac{M}{X}$ is noncosingular. Hence $M = X$. It follows that $N \ll M$.
(2) Let $N \ll_\gamma M$ and $N + X = M$. Then $\frac{M}{X}$ is noncosingular as M is noncosingular. Therefore by assumption $M = X$.
(3) Set $D \oplus D' = M$ and $N/D \ll M/D$. Then $N + D' = M$. As D is noncosingular and $N \ll_\gamma M$, we have $M = D'$. Therefore, $D = 0$ implying that $N \ll M$. □

Recall that a ring R is a right V-ring provided every simple right R-module is injective. It follows from [3, Proposition 2.5 and Corollary 2.6] that every right R-module over a right V-ring R is noncosingular. It follows from last proposition that the only γ-small submodule of an R-module over a right V-ring R is zero.

It is known that if $M \ll M$, then $M = 0$. But the following example shows that in γ-small case, it is not true.

Example 2.6. *Consider $\mathbb{Z}_4$ as a module over itself. Suppose $\mathbb{Z}_4 + X = \mathbb{Z}_4$ with $\frac{\mathbb{Z}_4}{X}$ noncosingular. We know $\overline{Z}(\mathbb{Z}_4) = \{0, 2\} \neq \mathbb{Z}_4$ and $\overline{Z}(\mathbb{Z}_4) \neq 0$. Hence the only noncosingular homomorphic image of $\mathbb{Z}_4$ is $\frac{\mathbb{Z}_4}{\mathbb{Z}_4}$. It follows by Proposition 2.3 that $\mathbb{Z}_4 \ll_\gamma \mathbb{Z}_4$.*

In contrast to small submodules, if N is a γ-small direct summand of M, then N need not be zero. Let M (which is decomposable) be a n-generated module ($n \geq 2$) over a Dedekind domain R. Consider a nontrivial decomposition $M_1 \oplus M_2 = M$ for the R-module M. Since every homomorphic image of M is cosingular, then $M_i \ll_\gamma M$. In fact, every submodule of M is γ-small in M by Proposition 2.3.

It is natural to consider an analogue of the sum of all small submodules in γ-case. Let M be a module. We define $\gamma(M)$ to be the sum of all γ-small submodules of M. It is clear that $Rad(M) \subseteq \gamma(M)$. Note also that, $Rad(M) = \gamma(M)$ holds for a noncosingular module M. The following contain examples of modules that shows the last inclusion is strict.

Example 2.7. (1) *It is clear that* $0 = Rad(\mathbb{Z}_{\mathbb{Z}}) \subset \gamma(\mathbb{Z}_{\mathbb{Z}}) = \mathbb{Z}$.

(2) *Let M be an amply supplemented module with $\overline{Z}^2(M) = 0$ (for example, M is cosingular). Then $\gamma(M) \neq 0$. In contrary, suppose $\gamma(M) = 0$. Then there is a proper submodule X of M such that $\frac{M}{X}$ is noncosingular. Now, $\frac{M}{X} = \overline{Z}^2(\frac{M}{X}) = \frac{\overline{Z}^2(M)+X}{X} = 0$, which is a contradiction. Therefore an amply supplemented module M with $\overline{Z}^2(M) = 0$ has at least a nonzero γ-small submodule. As a conclusion, for every $m \in \mathbb{N}$ we conclude that $\gamma(\mathbb{Z}_m) \neq 0$. In particular for $\mathbb{Z}_m$ as an $\mathbb{Z}$-module we have $Rad(\mathbb{Z}_m) \subset \gamma(\mathbb{Z}_m) = \mathbb{Z}_m$.*

Lemma 2.8. *Let N be a proper submodule of M with $\frac{M}{N}$ noncosingular. Let $x \in M \setminus N$ such that $Rx + N = M$. Then there is a maximal submodule K of M with $\frac{M}{K}$ noncosingular and $x \notin K$*

Proof. Set $\mathcal{A} = \{L \leq M \mid N \subseteq L, \frac{M}{L}$ is noncosingular, $x \notin L\}$. Then $\mathcal{A} \neq \varnothing$ since $N \in \mathcal{A}$. Suppose $\{L_\alpha\}$ is a chain in $\mathcal{A}$. We prove $\mathcal{A}$ has a maximal element. It is clear $\cup L_\alpha$ is a submodule of M and $N \subseteq \cup L_\alpha$. It is obvious that $x \notin \cup L_\alpha$. Note that $\frac{M}{\cup L_\alpha}$ is noncosingular as well as $\frac{M}{L_\alpha}$ for each α ($\frac{M}{\cup L_\alpha}$ is a homomorphic image of $\frac{M}{L_\alpha}$). Hence $\mathcal{A}$ has a maximal element say K. Now, suppose $K \subset T \subseteq M$ for a submodule T which properly contains K. Then $T \notin \mathcal{A}$ as K is the maximal element of $\mathcal{A}$. Hence $x \in T$. Therefore $M = Rx + N \subseteq T$. It shows that K is a maximal submodule of M. □

Last result leads us to find an equivalent set for $\gamma(M)$.

Theorem 2.9. *Let M be a module. Then $\gamma(M) = \bigcap\{N \leq_{max} M \mid \frac{M}{N}$ is noncosingular$\}$*

Proof. Let N be an arbitrary maximal submodule of M with $\frac{M}{N}$ noncosingular. Let also $K \ll_\gamma M$. Consider the submodule $N+K$ of M. If $N+K = M$, then $M = N$ as $K \ll_\gamma M$, which is a contradiction. Hence $N+K = N$, which implies $K \subseteq N$. So that $\sum_{K \ll_\gamma M} K \subseteq N$. Therefore $\sum_{K \ll_\gamma M} K \subseteq \bigcap\{N \mid N \leq_{max} M$ and $\frac{M}{N}$ is noncosingular$\}$. For the other side of inclusion, let $x \in \bigcap\{N \mid N \leq_{max} M$ and $\frac{M}{N}$ is noncosingular$\}=P$. Suppose that $xR + L = M$ with $\frac{M}{L}$ noncosingular. If $L \neq M$, then by Lemma 2.8, there is a maximal submodule K' of M with $\frac{M}{K'}$ noncosingular and $x \notin K'$. But $x \in P$ implies $x \in K'$, a contraction. Therefore $L = M$. So $xR \ll_\gamma M$, which implies $x \in \sum_{K \ll_\gamma M} K$. It follows that $P \subseteq \sum_{K \ll_\gamma M} K$, which completes the proof. □

Remark 2.10. *Let R be a ring and M be a right R-module. If $\mathcal{SN}$ denotes the class of all simple noncosingular (injective) right R-modules, then $\gamma(M) = Rej_M(\mathcal{SN})$.*

We are ready to consider $\gamma(R_R)$ for a ring R. By Theorem 2.9, we have $\gamma(R_R) = \bigcap\{I \leq R_R \mid R/I$ is simple injective $\}$.

Proposition 2.11. *Let R be a ring. Then $\gamma(R_R)$ is the largest γ-small right ideal of R.*

Proof. Let $\gamma(R_R)+I = R$ where R/I is noncosingular. Then there is a maximal right ideal I_0 of R such that $I \subseteq I_0$. Note that R/I_0 is noncosingular as well as R/I. By the definition of $\gamma(R)$ we conclude that $\gamma(R_R) \subseteq I_0$ which implies that $I_0 = R$, a contradiction. Therefore $I = R$, as required. □

Let R be a ring. Then R is said to be a *generalized V-ring (shortly GV-ring)* provided every simple singular right R-module is injective. In [4], the authors proved that R is right GV if and only if every simple cosingular right R-module is projective (see [4, Theorem 3.1]).

Proposition 2.12. *Let R be a ring. Then every simple right R-module is small (cosingular) if and only if $\gamma(R_R) = R$. In particular, if R is a right GV-ring and $\gamma(R_R) = R$, then R is a semisimple ring.*

Proof. If R is a right GV-ring and $\gamma(R_R) = R$, then every simple right R-module is projective by [4, Theorem 3.1]. Then R is semisimple. □

Let R be a commutative domain which is not a field. Then every finitely generated R-module is small and hence cosingular. Therefore, every simple R-module is small showing that $\gamma(R) = R$.

Example 2.13. *(see also [4, Example 3.15]) Let F be a field and let R be the ring of all upper triangular 2×2 matrices with entries from F. It is well-known that R is a left and right perfect GV-ring. Note that $J(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ which implies that R is not semisimple. Therefore $\gamma(R_R) \neq R$ and then there exists a simple injective right R-module.*

We should recall the definition of coclosed submodules of a module before presenting an analogue in γ-case. Let M be a module and N a submodule of M. Then N is called *coclosed* in case $N/K \ll M/K$ implies $N = K$. We say that N is a *γ-coclosed* submodule of M (denoted by $N \leq_{\gamma cc} M$) if $N/X \ll_\gamma M/X$ implies $N = X$. It follows by definitions that every γ-coclosed submodule of a module M is coclosed in M. But the converse may not hold. Let M be a finitely generated R-module, where R is a Dedekind domain which is not a field. Then any submodule of M is γ-small in M. If M has a nontrivial decomposition $M = \oplus_{i\in I} M_i$, then M_i is a coclosed submodule of M while non of nonzero submodules of M is γ-coclosed in M (for example consider the $\mathbb{Z}$-module $\mathbb{Z}_n$ where n is square-free).

It is not hard to verify that for a noncosingular module the two concepts coclosed and γ-coclosed are the same.

The following contains some properties of γ-coclosed submodules of a module.

Proposition 2.14. *Let M be an R-module and let $A \leq B \leq M$. Then:*

(1) *Let $A \leq_{\gamma cc} M$. If $X \leq A \leq M$ and $X \ll_\gamma M$, then $X \ll_\gamma A$.*

(2) *If B is γ-coclosed in M, then $\frac{B}{A}$ is γ-coclosed in $\frac{M}{A}$.*

(3) *If A is γ-coclosed in M, then A is γ-coclosed in B. The converse holds, in case B is γ-coclosed in M.*

(4) *Let C be a γ-coclosed submodule of M, then for any $A \leq B \leq C$, $\frac{B}{A} \ll_\gamma \frac{M}{A}$ if and only if $\frac{B}{A} \ll_\gamma \frac{C}{A}$.*

Proof. (1) Suppose that A is a γ-coclosed submodule of M and $X \ll_\gamma M$. To show that $X \ll_\gamma A$, let $A = X + K$ such that $\frac{A}{K}$ is noncosingular. Since A is γ-coclosed in M, it is sufficient to show that $\frac{A}{K} \ll_\gamma \frac{M}{K}$. Let $\frac{M}{K} = \frac{A}{K} + \frac{B}{K}$ where $\frac{M}{B}$ is noncosingular. Then $M = A + B = X + K + B = X + B$. But $X \ll_\gamma M$ combining with being $\frac{M}{B}$ noncosingular implies $M = B$. So we get the result.

(2) Assume that B is γ-coclosed in M. Let $\frac{B/A}{X/A} \ll_\gamma \frac{M/A}{X/A}$ where $A \leq X \leq B \leq M$. We show that $B/X \ll_\gamma M/X$. To verify the last assertion, suppose $B/X + T/X = M/X$ with M/T noncosingular. Then $\frac{B/A}{X/A} + \frac{(T+A)/A}{X/A} = \frac{M/A}{X/A}$. Note that $M/(T+A)$ as a homomorphic image of M/T is noncosingular. Now, since $\frac{B/A}{X/A} \ll_\gamma \frac{M/A}{X/A}$, we conclude that $B/X \ll_\gamma M/X$. The fact that B is γ-coclosed in M, implies $B = X$. Hence $B/A = X/A$. It follows that B/A is a γ-coclosed submodule of M/A.

(3) Suppose that A is γ-coclosed in M and let X be a submodule of A such that $\frac{A}{X} \ll_\gamma \frac{B}{X}$. Then by Proposition 2.2, $\frac{A}{X} \ll_\gamma \frac{M}{X}$. Being A a γ-coclosed submodule of M implies $A = X$. Thus A is γ-coclosed in B. For the converse, assume that A is γ-coclosed in B and B is γ-coclosed in M. Let X be a submodule of A such that $\frac{A}{X} \ll_\gamma \frac{M}{X}$. Then by (2) we have $\frac{B}{X} \leq_{\gamma cc} \frac{M}{X}$. Now by (1) we conclude that $\frac{A}{X} \ll_\gamma \frac{B}{X}$. Since A is γ-coclosed in B, we must have $A = X$. Thus A is γ-coclosed in M.

(4) Let $B/A \ll_\gamma C/A$. Then by Proposition 2.2(2), $B/A \ll_\gamma M/A$. For the converse, let $B/A \ll_\gamma M/A$. Since $C \leq_{\gamma cc} M$, by (2) we have $C/A \ll_{\gamma cc} M/A$. Now the result follows from (1).

□

Definition 2.15. Let M be a module. Then we call M, γ-*hollow* provided every proper submodule of M is γ-small in M.

Note that every finitely generated $\mathbb{Z}$-module is γ-hollow while it might not be a hollow module.

Clearly every hollow module is γ-hollow. Note that if M is a noncosingular module, then M is hollow if and only if M is γ-hollow.

Proposition 2.16. *Let M be an R-module. Then M is γ-hollow if and only if every proper submodule A of M with M/A noncosingular, is small in M.*

Proof. ($\Rightarrow$) Let A be a proper submodule of M such that $\frac{M}{A}$ is noncosingular. We show that $A \ll M$. Assume that there exists $B \subset M$ such that $M = A+B$. Since M is γ-hollow, then $B \ll_\gamma M$. As $\frac{M}{A}$ is noncosingular, then $M = A$ which contradicts $A < M$. Thus $B = M$.

($\Leftarrow$) To show that M is γ-hollow, let A be a proper submodule of M. To contrary, assume that A is not γ-small in M which means that there exists a proper submodule B of M such that $\frac{M}{B}$ is noncosingular and $M = A + B$. Being B a small submodule of M implies $A = M$. □

Proposition 2.17. *A nonzero homomorphic image of a γ-hollow module is γ-hollow.*

Proof. Let M be a γ-hollow module and $N < M$. Assume that L/N is a proper submodule of M/N. Then L is a proper submodule of M which is γ-small in M. Hence $L/N \ll_\gamma M/N$ by Proposition 2.2. □

3 γ-lifting modules

In this section we shall introduce a new generalization of lifting modules via γ-small submodules.

Definition 3.1. Let M be a module. We say M is *γ-lifting* provided for every submodule N of M there exists a direct summand D of M contained in N such that $N/D \ll_\gamma M/D$.

It is obvious that every lifting module is γ-lifting.

Example 3.2. (1) *Every γ-hollow module is γ-lifting.*

(2) *Every lifting module is γ-lifting. But the converse does not hold. Consider the $\mathbb{Z}$-module $\mathbb{Z}$. As $\mathbb{Z}$ is γ-hollow, it is γ-lifting. Note that $\mathbb{Z}$ is not a lifting module.*

(3) *For a noncosingular module, two concepts lifting and γ-lifting coincide.*

We can verify the following easily.

Proposition 3.3. *Let M be an indecomposable module. Then M is γ-lifting if and only if M is γ-hollow.*

Theorem 3.4. *Let M be a module. Then the following are equivalent:*

(1) *M is γ-lifting;*

(2) *For every submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq N$ and $N \cap M_2 \ll_\gamma M_2$;*

(3) *Every submodule N of M can be written as a direct sum of a direct summand K of M and a γ-small submodule L of M.*

Proof. (1) $\Rightarrow$ (2) Let M be γ-lifting and $N \leq M$. Then there is a direct summand D of M contained in N such that $N/D \ll_\gamma M/D$. Set $M = D \oplus D'$ for some submodule D' of M. Let $(D' \cap N) + T = D'$ with D'/T noncosingular. Then $N + T = N + D' = M$. It follows now that $N/D + (T + D)/D = M/D$. Since $(D' \cap N) + T = D'$ we have $(D' \cap N) + T + D = M$. So $\frac{M}{T+D} \cong \frac{D' \cap N}{(D' \cap N) \cap (T+D)} = \frac{D' \cap N}{T \cap N}$. As $\frac{D' \cap N}{T \cap N} \cong \frac{D'}{T}$ and $\frac{D'}{T}$ is noncosingular, we conclude that $\frac{M}{D} = \frac{T+D}{D}$. Hence $M = T + D$. Now the modular law implies $T = D'$.

(2) $\Rightarrow$ (3) Let $N \leq M$. then by (2) there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll_\gamma M_2$. By modularity we have $N = M_1 \oplus (N \cap M_2)$.

(3) $\Rightarrow$ (1) Conversely, let N be a submodule of M. Then there exists a decomposition $N = K \oplus L$ with K a direct summand of M and $L \ll_\gamma M$. We shall verify $N/K \ll_\gamma M/K$. Suppose that $N/K + T/K = M/K$ such that M/T is noncosingular. Then $N + T = M$. So $L + T = M$. Since $L \ll_\gamma M$, we conclude that $T = M$. Therefore, $N/K \ll_\gamma M/K$. □

Proposition 3.5. *Every supplement submodule of a γ-lifting module is γ-lifting.*

Proof. Let M be a γ-lifting module and let N be a supplement of a submodule K in M. Then $M = N + K$ and $N \cap K \ll N$. Suppose that L is a submodule of N. Being M, γ-lifting there is a direct summand D of M such that $L/D \ll_\gamma M/D$. We shall prove that $L/D \ll_\gamma N/D$. To verify it, let $L/D + H/D = N/D$ with N/H noncosingular. Then $L + H = N$ which implies $L + H + K = M$. So, $L/D + (H + K)/D = M/D$. Consider $M/(H + K)$ which is isomorphic to $\frac{L}{L \cap (H+K)}$. Note that $\frac{L}{L \cap (H+K)}$ is a homomorphic image of the noncosingular module $L/(L \cap H) \cong N/H$. Now, we conclude that $H + K = M$. Modularity implies $H + (N \cap K) = N$ and $N \cap K \ll N$ yields $H = N$ as required. □

Proposition 3.6. *Let M be a γ-lifting module and N a submodule of M. If for every direct summand D of M, the submodule $(D+N)/N$ is a direct summand of M/N, then M/N is γ-lifting.*

Proof. Let L/N be a submodule of M/N. Then there exists a direct summand D of M contained in L such that $L/D \ll_\gamma M/D$. We shall show that $\frac{L/N}{(D+N)/N} \ll_\gamma \frac{M/N}{(D+N)/N}$. To verify the assertion, suppose $\frac{L/N}{(D+N)/N} + \frac{T/N}{(D+N)/N} = \frac{M/N}{(D+N)/N}$ for a submodule T of M which contains $D + N$ such that $\frac{\frac{M/N}{(D+N)/N}}{\frac{T/N}{(D+N)/N}}$ is noncosingular which implies M/T is a noncosingular module. Then $L/N + T/N = M/N$. So that $L/D + T/D = M/D$ where M/T is noncosingular. As L/D is a γ-small submodule of M/D, we conclude that $M/D = T/D$ that completes the proof. Note also that by assumption $(D + N)/N$ is a direct summand of M/N. □

Corollary 3.7. *Let M be a module N a submodule of M such that for every decomposition $M = M_1 \oplus M_2$ we have $N = (N \cap M_1) \oplus (N \cap M_2)$. If M is γ-lifting, then M/N is γ-lifting.*

Proof. Let D be a direct summand of M. Set $D \oplus D' = M$. Then $(D+N)/N + (D'+N)/N = M/N$. Note that $N = (N \cap D) \oplus (N \cap D')$. It is not hard to verify that $(D+N) \cap (D'+N) = N$ which shows that $(N+D)/N \oplus (N+D')/N = M/N$. Hence M/N is γ-lifting by Proposition 3.6. □

Corollary 3.8. *Let M be a module and N a projection invariant (fully invariant) submodule of M. If M is γ-lifting, then M/N is γ-lifting. In particular, every homomorphic image of a duo (distributive) γ-lifting module is γ-lifting.*

Proposition 3.9. *Let M be a γ-lifting module. Then $M/\gamma(M)$ is semisimple.*

Proof. Let $N/\gamma(M)$ be an arbitrary submodule of $M/\gamma(M)$. Then there is a decomposition $M = D \oplus D'$ with $D \subseteq N$ and $N \cap D' \ll_\gamma D'$. It follows that $N/\gamma(M) + (D' + \gamma(M))/\gamma(M) = M/\gamma(M)$. Note that $N \cap (D' + \gamma(M)) = \gamma(M) + (N \cap D')$. As $N \cap D' \ll_\gamma D'$, it must be contained in $\gamma(M)$ by Remark 2.10. Hence $M/\gamma(M)$ is semisimple. □

Corollary 3.10. *Let M be a γ-lifting module with $\gamma(M) = 0$. Then M is semisimple. In particular, over a right V-ring R a right R-module M is γ-lifting if and only if M is semisimple.*

The next example shows that a direct sum of γ-lifting modules may not be γ-lifting.

Example 3.11. (1) *Let p be a prime integer and let $n \geq 1$ be an integer. Consider the $\mathbb{Z}$-modules $M_1 = \oplus_{i=1}^n M_i$ and $M_2 = \oplus_{i \in \mathbb{N}} M_i$, where $M_i = \mathbb{Z}(p^\infty)$ for all $i \in \mathbb{N}$. It is clear that M_1 and M_2 are noncosingular. By [2, Propositions A.7 and A.8], the module M_1 is lifting but M_2 is not lifting. So M_1 is γ-lifting but M_2 is not γ-lifting.*

(2) *Let R be an incomplete rank one discrete valuation ring with quotient field K. Consider the R-module $M = K^2$. Clearly, M is a noncosingular module. By [2, Lemma A.5], the module M is not amply supplemented. So M is not lifting. Therefore M is not a γ-lifting module. On the other hand, the R-module K is γ-lifting since it is lifting (see [2, Proposition A.7]).*

Theorem 3.12. *Let $M = M_1 \oplus M_2$ be a duo module. Then M is γ-lifting if and only if M_1 and M_2 are γ-lifting.*

Proof. Let $M = M_1 \oplus M_2$ be a duo module such that M_1 and M_2 are γ-lifting. Suppose that N is an arbitrary submodule of M. Then $N = (N \cap M_1) \oplus (N \cap M_2)$. Since M_1 and M_2 are γ-lifting, there exist direct decompositions $N \cap M_1 = K_1 \oplus L_1$ and $N \cap M_2 = K_2 \oplus L_2$ such that K_i is a direct summand of M_i for $i = 1, 2$ and L_i is a γ-small submodule of M_i for $i = 1, 2$ by Theorem 3.4. Then $N = (N \cap M_1) \oplus (N \cap M_2) = (K_1 \oplus K_2) \oplus (L_1 \oplus L_2)$. It is obvious that $K_1 \oplus K_2$ is a direct summand of M. Note that $L_1 \oplus L_2$ is a γ-small submodule of M by Proposition 2.2(4). Hence M is a γ-lifting module by Theorem 3.4. The converse follows from Proposition 3.5. $\square$

The following provides a special decomposition of γ-lifting modules.

Proposition 3.13. *Let M be an amply supplemented γ-lifting module. Then $M = \overline{Z}^2(M) \oplus M'$ where $\overline{Z}^2(M)$ is a noncosingular lifting module.*

Proof. Assume that M is a γ-lifting module. By Theorem 3.4, there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$, $M_1 \subseteq \overline{Z}^2(M)$ and $\overline{Z}^2(M) \cap M_2 \ll_\gamma M_2$. Therefore $\overline{Z}^2(M) = M_1 \oplus (\overline{Z}^2(M) \cap M_2)$. This implies that $\overline{Z}^2(M) \cap M_2 = \overline{Z}^2(M_2)$ is a small submodule of M by Proposition 2.5. As M is amply supplemented, we conclude that $\overline{Z}^2(M_2)$ is a noncosingular submodule of M. This yields that $\overline{Z}^2(M_2) = 0$ and $\overline{Z}^2(M) = M_1$. Thus $M = \overline{Z}^2(M) \oplus M_2$. Note that, since $\overline{Z}^2(M)$ is γ-lifting and noncosingular , it should be lifting. $\square$

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