East-West J. of Mathematics: Vol. 22, No 1 (2020) pp. 95-101
https://doi.org/10.36853/ewjm0371

ON FULLY BOUNDED NOETHERIAN MODULES AND THEIR ENDOMORPHISM RINGS

Nguyen Van Sanh*, **Oravan Arunphalungsanti**
Nguyen Trong Bac and **Sarapee Chairat**†

Dept. of Mathematics, Faculty of Science
Mahidol University, Bangkok 10400, Thailand,
Center of Exellence in Mathematics,
The Commission on Higher Education, Thailand, and
Nguyen Tat Thanh Univ., HoChiMinh city, Vietnam
e-mail: nguyen.san@mahidol.ac.th;oravan@mutacth.com; bacnt2008@gmail.com.

†*Dept. of Mathematics, Faculty of Science*
Thaksin University, Patthalung 93110, Thailand,
e-mail: sarapee@tsu.ac.th

Abstract

In this paper we introduce the notion of bounded modules and fully bounded modules. A right R-module M is called a bounded module if every essential submodule of M contains a fully invariant submodule of M which is essential in M_R. A module M is called a fully bounded module if M/X is bounded for any prime submodule X of M.

1. Introduction and Preliminaries

Throughout this paper, all rings are associative with identity, and all modules are unitary right R-modules. We write M_R (resp. ${}_RM$) to indicate that M is a right (resp. left) R-modules. We also write $J(R)$ (resp. $rad(M)$) for the Jacobson radical of R (resp. Jacobson radical of M_R) and $S = End(M_R)$, its endomorphism ring. A submodule X of M is called a *fully invariant* submodule

*Correspoding author.
Key words: bounded modules, fully bounded module, Jacobson's conjecture.
2010 AMS Mathematics Classification: 16D50,16D70,16D80.

of M if for any $f \in S$, we have $f(X) \subset X$. Especially, a right ideal of R is a fully invariant submodule of R_R if it is a two-sided ideal of R. Following [16], a fully invariant proper submodule X of M is called *prime submodule* of M if for any ideal I of S and any fully invariant submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. In particular, an ideal P of R is a prime ideal if for any ideals I, J of R, if $IJ \subset P$, then either $I \subset P$ or $J \subset P$. A non-zero submodule U of M is called *essential* in M if U has non-zero intersection with any non-zero submodule of M. A right R-module M is called a *self-generator* if it generates all its submodules. M is *retractable* if for any non-zero submodule X of M, there is a non-zero $\varphi \in S = \mathrm{End}(M)$ such that $\varphi(M) \subset X$. Clearly, every self-genarator is retractable. Note that, for a submodule X of M, if M is retractable and $\mathrm{Hom}(M, X) = 0$, then $X = 0$. General background materials can be found in [1], [3], [4], [5], [6], [10], [11], [13], [18], [19].

2. Bounded and fully bounded modules

Definition 1.1 A right R-module M is called a *bounded module* if every essential submodule contains a fully invariant submodule which is essential in M as a submodule. A ring R is a right bounded if every essential right ideal of R contains an ideal which is essential as a right ideal.

Clearly, every commutative ring is right bounded. A simple Artinian ring which has no proper essential right ideals is right bounded.

Let X be a submodule of M. We denote $I_X = \{f \in S | f(M) \subset X\}$. Clearly, I_X is a right ideal of S. If X is a fully invariant submodule of M, then I_X is an ideal of S. The following two properties will be useful.

Prposition 2.2 *Let M be a quasi-projective finitely generated right R-module which is retractable.If X is an essential submodule of M, then I_X is an essential right ideal of $S = End_R(M)$.*

The proof is similar to ([14], Lemma 3.6) with notice that if M is retractable and $\mathrm{Hom}(M, X) = 0$, then $X = 0$.

Proposition 2.3 *Let M be a quasi-projective, finitely generated right R-module which is retractable. If K is an essential right ideal of S, then $K(M)$ is an essential submodule of M.*

Proof. Suppose that $K(M) \cap B = 0$ with B is a submodule of M. Then we have $Hom(M, K(M) \cap B) = Hom(M, K(M)) \cap Hom(M, B) = 0$. By ([19, 18.4]) we have $K = Hom(M, K(M))$. Since M is retractable and $Hom(M, B) = 0$, we can see that $B = 0$. Hence $K(M)$ is an essential submodule of M. □

From above propositions, we have the following theorem.

Theorem 2.4 *Let M_R be a quasi-projective, finitely generated right R-module which is retractable. Then, M_R is a bounded module if and only if its endomorphism ring $S = End(M_R)$ is a right bounded ring.*

Proof. Suppose that M is a bounded module. Let I be an essential right ideal of S. Then $I(M)$ is essential submodule of M. By assumption, $I(M)$ contains a fully invariant submodule B of M which is essential in M. By Proposition 2.2 the ideal I_B is an essential right ideal of S. Note that $I_B \subset I_{I(M)} = I$ by [19, 18.4] and thus S is a right bounded ring.

Conversely, assume that S is a right bounded ring. Let X be an essential submodule of M. By proposition 2.2, I_X is an essential right ideal of S. By hypothesis, there exists a two sided ideal K of S contained in I_X which is essential in S as a right ideal. Note that $K(M)$ is fully invariant submodule of M. By proposition 2.3, $K(M)$ is an essential submodule of M_R, proving that M is a bounded module. □

Lemma 2.5 ([19, 17.3]) *Let K, M, N be R-modules. If $f : M \to N$ is a homomorphism and K is an essential submodule of N, then $f^{-1}(K)$ is an essential submodule of M.*

Theorem 2.6 *If M is a bounded module, then so is M^n for any $n \in \mathbb{N}$.*

Proof. Suppose that M is a bounded module and X is any essential submodule of M^n. Write $M^n = \oplus_{i=1}^n M_i$, where $M_i = M$ for each $i = 1, 2, ..., n$. Then $X \cap M_i$ is essential in M_i for each $i = 1, 2, ..., n$. By assumption, $X \cap M_i$ contains a fully invariant submodule A_i of M_i such that A_i is essential in $M_i = M$, $i = 1, 2, ..., n$. Put $B = \bigcap_{i=1}^n A_i$. Then B is a fully invariant submodule of M_i $(i = 1, 2, ..., n)$. Hence $B^n = \oplus_{i=1}^n B_i, B_i = B$, is essential in M^n. It remains to prove that B^n is a fully invariant submodule of M^n. Let $\varphi \in End(M^n)$. Then $\varphi = (\varphi_{ij})$, $\varphi_{ij} : M_j \to M_i = M$ with $\varphi_{ij} = \pi_i \varphi \iota_j \in End(M)$, where $\iota_j : M_j \to M^n, \pi_i : M^n \to M_i$ are inclusion and projection maps. Take any $x = (b_1, ..., b_n) \in B^n = \oplus_{i=1}^n B_i$. Then $x = \sum_{j=1}^n \iota_j(b_j)$ and therefore $\varphi(x) = \sum_{i=1}^n \varphi \iota_j(b_j) = \sum_{i=1}^n \iota_i \pi_i(\sum \varphi \iota_j(b_j))$. Thus $\varphi(x) = \sum_{i=1}^n \iota_i[\sum_{i=1}^n \pi_i \varphi \iota_j(b_j)] = \sum_{i=1}^n \iota_i[\sum_{i=1}^n \varphi_{ij}(b_j)]$, where $b_j \in B_j = B$ and hence $\varphi_{ij}(b_j) \in B_i = B$. Therefore $\sum \varphi_{ij}(b_j) \in B_i \subset M_i$. Hence $\sum_{i=1}^n \iota_i[\sum_{i=1}^n \varphi_{ij}(b_j)] \in B^n$, proving that B^n is a fully invariant submodule of M^n. □

Let $Mat_n(R)$ be the ring of all square matrices of order n with coefficients in R. The following corollary is an immediate consequence.

Corollary 2.7 *If R is a right bounded ring, then R^n is a bounded R-module and hence $Mat_n(R)$ is a right bounded ring.*

Lemma 2.8 ([16], Theorem 2.4) *Let M be a right R-module. If M is a prime R-module, then its endomorphism ring S is a prime ring. Conversely, if M is a self-generator and S is a prime ring, then M is a prime module.*

It was showed in [6] that a prime ring R is right bounded if and only if every essential right ideal of R contains a non-zero ideal. Using this result, we have the following theorem.

Theorem 2.9 *Let M be a quasi-projective, finitely generated right R-module which is retractable. If M is a prime module, then M is a bounded module if and only if every essential submodule of M contains a non-zero fully invariant submodule of M.*

Proof. One way is clear by definition. Conversely, let I be an essential right ideal of S. Then $I(M)$ is an essential submodule of M. By assumption, $I(M)$ contains a fully invariant submodule B of M. Since $I(M)$ is an essential submodule of M and $0 \neq B \subset_{>} I(M)$. By 2.3, $I_B \subset I_{I(M)} = I$ and I_B is an ideal of S. Since M is a prime module,it follows that S is a prime ring, by Lemma 2.8. Therefore, S is a right bounded ring. It follows from Theorem 2.4 that M is a bounded module. □

Recall that a ring R is right fully bounded if for every prime ideal I of R, the prime factor ring R/I is a right bounded ring. We now introduce the concept of fully bounded modules as a generalization of fully bounded rings.

Definition 2.10 A right R-module M_R is fully bounded if for every prime submodule X of M, the factor module M/X is a bounded module. A ring R is right fully bounded if for every prime ideal I of R, the factor ring R/I is a right bounded ring.

We now examine the relationship between a fully bounded module M and its endomorphism ring S. First we need the following lemmas, the proofs which are straightforward.

Lemma 2.11 *Let X be a fully invariant submodule of M, $\varphi = End(M)$. Then there is a unique $\bar{\varphi} = End(M/X)$ such that $\bar{\varphi}\nu = \nu\varphi$, where $\nu : M \to M/X$ is the natural projection.*

Lemma 2.12 *Let X be a submodule of a quasi-projective module M, $\psi \in End(M/X)$. There is a $\varphi \in End(M)$ such that $\psi\nu = \nu\varphi$ where $\nu : M \to M/X$.*

Lemma 2.13 *Let M be a quasi-projective right R-module and X, a fully invariant submodule of M. Then $End(M/X) \simeq S/I_X$, where $S = End(M)$ and $I_X = \{\varphi | \varphi(M) \subset X\}$.*

Proof. Let $\varphi \in S$ and $\bar{\varphi} \in \bar{S} = End(M/X)$ as defined in Lemma 2.11 and 2.12. Define the map $\Phi : End(M/X) \to S/I_X$ given by $\bar{\varphi} \mapsto \varphi + I_X$. Clearly, Φ is well-defined. Note that for $\psi, \varphi \in S$, we have $\bar{\varphi} + \bar{\psi} = \overline{\varphi + \psi}$ and $\bar{\varphi} \cdot \bar{\psi} = \overline{\varphi \cdot \psi}$. Using these facts we can check that Φ is a ring homomorphism. Moreover, it can be seen that Φ is 1-1 and onto, proving that Φ is a ring isomorphism. □

Lemma 2.14 *Let X be a fully inveriant sub module of M.*

(1) If M is quasi-projective, then so is M/X.

(2) If M is retractable, then so is M/X.

(3) If M is a self-generator, then so is M/X.

Proof.

(1) Let $g : M/X \to N$ be any R-epimorphism and $h : M/X \to N$. Then $g\nu$ is an R-epimorphism. Since M is quasi-projective, there exists $\varphi \in End(M)$ such that $(g\nu)\varphi = h\nu$. Since X is fully invariant, there is a unique $\bar{\varphi} \in End(M/X)$ such that $\bar{\varphi}\nu = \nu\varphi$. Hence $h\nu = g\nu\varphi = g\bar{\varphi}\nu$. It follows that $h = g\bar{\varphi}$, proving that M/X is quasi-projective.

(2) Let B be any submodule of M/X. Then B is of the form A/X for some submodule A of M.

If M is retractable, then there is $\varphi \in S$ such that $\varphi(M) \subset A$. Since $\varphi(M)/X \simeq \overline{\varphi}(M/X)$, we get $\overline{\varphi}(M/X) \subset B$, proving that M/X is retractable.

(3) If M is a self-generator, $A = \Sigma_{\varphi \in I}\varphi(M)$ for some subset I of S. Applying Lemma 2.11 and 2.12, $\varphi(M)/X \simeq \bar{\varphi}(M/X)$ and hence $B = \Sigma_{\varphi \in I}\bar{\varphi}(M/X)$, proving that M/X is a self-generator. □

Lemma 2.15 ([16], Theorem 1.10) *Let M be a right R-module, $S = End(M_R)$ and X, a fully invariant submodule of M. If X is a prime submodule of M, then I_X is a prime ideal of S. Conversely, if M is a self-generator and if I_X is a prime ideal of S, then X is a prime submodule of M.*

Theorem 2.16 *Let M be a quasi-projective, finitely generated right R-module which is self-generator. Then M is a fully bounded module if and only if S is a right fully bounded ring.*

Proof. Let I be any prime ideal of S. Then $X = I(M)$ is a fully invariant submodule of M. Note that $I = Hom(M, I(M))$ by [20, 18.4] and hence $X \neq M$ and $I_X = Hom(M, I_X(M)) = Hom(M, X) = Hom(M, I(M)) = I$. This shows that X is a prime submodule of M by Lemma 2.14. By assumption, M/X is a bounded module. It follows from Theorem 2.4 that End(M/X) is a right bounded ring. By Lemma 2.13, $S/I = S/I_X \simeq End(M/X)$ is a right bounded ring, proving that S is a right fully bounded ring.

Conversely, let X be a prime submodule of M. Then I_X is a prime ideal of S by Lemma 2.15. By assumption, S/I_X is a right bounded ring. Hence M/X is a bounded module, by Lemma 2.13. This shows that M is a fully bounded module and the proof of our Theorem is complete. □

Theorem 2.17 *Let M be a quasi-projective, finitely generated right R-module. If M is a Noetherian module, then S is a right Noetherian ring.*

Proof. Suppose that we have ascending chain of right ideals of S, $I_1 \subset I_2 \subset \cdots$. Then we have $I_1(M) \subset I_2(M) \subset \cdots$ is an ascending chain of submodules of M. By assumption, there exists an integer n such that $I_n(M) = I_k(M)$, for all $k > n$. By ([19, 18.4]), we have $I_n = Hom(M; I_n(M)) = Hom(M; I_k(M)) = I_k$. Thus S is a right Noetherian ring. □

A right fully bounded ring needs not be right bounded. However, if R is a right fully bounded right Noetherian ring, it can be shown that R is right bounded (see [3, Proposition 7.12]). Applying this Proposition, we can generalize the result to modules as follows.

Theorem 2.18 *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M_R is a Noetherian fully bounded module, then M_R is a bounded module.*

Proof. Since M_R is a Noetherian fully bounded module, S is a right Noetherian right fully bounded ring, by theorem 2.16 and 2.17. By [3, Proposition 7.12], S is a right bounded ring. By Theorem 2.4, we can see that M_R is a bounded module, completing our proof. □

Following [6], if R is a right Noetherian right fully bounded ring, then every factor ring of R is right bounded. Combining this result and Lemma 2.14, we can prove the following theorem.

Theorem 2.19 *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a fully bounded Noetherian module and X is a fully invariant submodule of M, then M/X is a bounded module.*

Proof. Since M is a right Noetherian right fully bounded module, the endomorphism ring S is right Noetherian, right fully bounded by Theorem 2.16 and 2.17. Let X be a fully invariant submodule of M. Then S/I_X is a right bounded ring by [6]. Let B/X be any essential submodule of M/X. Then by Lemma 2.13 and 2.14, I_B/I_X is an essential right ideal of S/I_X. Since S/I_X is a right bounded ring, I_B/I_X contains an essential ideal H/I_X of S/I_X. Therefore $H(M)/X$ is a fully invariant essential submodule of M/X, proving that factor module M/X is a bounded module. □

The following corollary is a direct consequence of the above theorem.

Corollary 2.20 *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and $f : M \to N$ be an epimorphism. If $Kerf$ is a* *fully invariant submodule of M and if M is a fully bounded Noetherian module, then N is a Noetherian bounded module.*

References

[1] Anderson F.W., Fuller K.R., *Rings and Categories of Modules.* Graduate Texts in Mathematics. No.13. Springer-Verlag. New York. Heidelberg. Berlin; 1974.

[2] Cauchon, *Sur l'intersection des puissance du radical d'un T-anneau Noetherian.* C. R. Acad. Sci. Paris,t.279(1974), Series A, 91-93.

[3] Chatters A.W. and Hajanavis C.R., "Rings with chain conditions", Pitman, London, 1980.

[4] Faith C., "Algebra I : Rings, Modules and Categories", Springer-Verlag, 1980.

[5] Faith C., "Algebra II. Ring Theory", Springer-Verlag, 1976.

[6] Goodearl K.R., Warfield R.B Jr. , "An Introduction to Noncommutative Noetherian Rings", (London Mathematical Society Student Texts No. 16). Cambridge University Press, 1989.

[7] Herstein I.N., *A Counterexample in Noetherian rings,* Proc. N. A. S., 54(1965), 1036-1037.

[8] Jacobson N., "The Structure of Rings", American Mathematics Society, Providence, 1956.

[9] Jategaonkar A.V., *Jacobson's conjecture and modules over fully bounded Noetherian rings.*, J. of Algebra,30(1974), 103-121.

[10] Kasch F. "Modules and Rings", Academic Press, London, New York, 1982.

[11] Lam T.Y., "Lectures on Modules and Rings", Graduate Texts in Mathematics, No.189, Springer-Verlag, 1998.

[12] Lenagan T.H., *Noetherian Rings with Krull Dimension One,* 2(15)(1977), 41-47.

[13] McConnell J.C. and Robson J.C., "Noncommutative Noetherian Rings", Graduate Studies in Mathematics 30. Amer. Math. Soc., Providence, Rhode Island, 2001.

[14] Sanh N.V., Vu N.A., Ahmed K.F.U., Asawasamrit S. and Thao L.P., *Primeness in module category.*, Asian-European Journal of Mathematics, 3(1)(2010), 145-154.

[15] Sanh N.V., Asawasamrit S., Ahmed K.F.U. and Thao L.P., *On prime and semiprime Goldie modules*, Asian-European Journal of Mathematics, **3**(2)(2011), 321-334.

[16] Sanh N.V. and Thao L.P., *A Generalization of Hopkins-Levitzki Theorem,* Southeast Asian Bulletin of Mathematics, **37**(4) (2013),591-600.

[17] N. V. Sanh and S. Sanpinij, *On Serial Artinian Modules and Their Endomorphism Rings,* Southeast Asian Bull. of Mathematics, **37**(3) (2013), 401-404

[18] Stenström B., "Rings of Quotients", Springer-Verlag, Berlin Heidelberg-New York, 1975.

[19] Wisbauer R., "Foundations of Module and Ring Theory", Gordon and Breach, Tokyo, e.a., 1991.