East-West J. of Mathematics: Vol. 23, No 2 (2022) pp. 119-137
https://doi.org/10.36853/ewjm0390

ALL MAXIMAL UNIT-REGULAR SIBMONOIDS OF RELHYP((2),(2))

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Abstract

Relational hypersubstitutions for algebraic systems are mappings which map operation symbols to terms and map relation symbols to relational terms preserving arities. The set of all relational hypersubstitutions for algebraic systems ($Relhyp(\tau,\tau')$) together with a binary operation defined on this set forms a monoid. In this paper, we determine all maximal unit-regular submonoids of this monoid of type $((2),(2))$.

Introduction

In universal algebra, identities are used to classify algebras into collections called *varieties* and *hyperidentities* are used to classify varieties into collections called *hypervarieties*[9]. The tool which is used to study hyperidentities and hypervarieties is the concept of a hypersubstitution. The notation of hypersubstitutions was introduced by K. Denecke et al. [2]. To recall the concept of a hypersubstitution of type τ, we recall first the concept of an m-ary term of type τ. Let $(f_i)_{i\in I}$ be a set of m_i-ary operation symbols indexed by the set I where $m_i \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. The set $X := \{x_1, \ldots, x_n, \ldots\}$ is a countably infinite set

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Key words: Hypersubstitution, relational hypersubstitution, algebraic systems
2010 AMS Classification: 20M07, 08B15, 08B25.

of symbols called *variables*. For each $m \geq 1$, let $X_m := \{x_1, \ldots, x_m\}$. We call the sequence $\tau := (m_i)_{i \in I}$ of arities of f_i, the type. An *m-ary term of type* τ is defined inductively as the following steps.

(i) Every variable $x_k \in X_m$ is an m-ary term of type τ.

(ii) If $t_1, \ldots, t_{m_i}$ are m-ary terms of type τ and f_i is an m_i-ary operation symbol, then $f_i(t_1, \ldots, t_{m_i})$ is an m-ary term of type τ.

Let $W_\tau(X_m)$ be the set of all m-ary terms of type τ which contains $x_1, \ldots, x_m$ and is closed under finite application of (ii) and let $W_\tau(X) := \bigcup_{m \in \mathbb{N}^+} W_\tau(X_m)$ be the set of all terms of type τ.

Arity is the number of arguments or operands taken by a function or operation and $\tau = (m_i)_{i \in I}$ be a type. A hypersubstitution of type τ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ preserving the arity. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. To define a binary operation on this set, we define inductively the concept of a superposition of terms $S_n^m : W_\tau(X_m) \times (W_\tau(X_n))^m \to W_\tau(X_n)$ by the following steps.

(i) If $t = x_k$ for $1 \leq k \leq m$, then $S_n^m(x_k, s_1, \ldots, s_m) := s_k$.

(ii) If $t = f_i(t_1, \ldots, t_{m_i})$, then
$S_n^m(t, s_1, \ldots, s_m) := f_i(S_n^m(t_1, s_1, \ldots, s_m), \ldots, S_n^m(t_{m_i}, s_1, \ldots, s_m))$.

For every $\sigma \in Hyp(\tau)$, we define a mapping $\widehat{\sigma} : W_\tau(X_m) \to W_\tau(X_m)$ as follows:

(i) $\widehat{\sigma}[x_k] := x_k \in X_m$,

(ii) $\widehat{\sigma}[f_i(t_1 \ldots, t_{m_i})] := S_m^{m_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{m_i}])$, for any m_i-ary operation symbol f_i and $\widehat{\sigma}[t_j]$ are already defined for all $1 \leq j \leq m_i$.

Further, a binary operation $\circ_h$ on the set $Hyp(\tau)$ is defined by $\sigma \circ_h \alpha = \hat{\sigma} \circ \alpha$, where $\circ$ denotes the usual composition of mappings. Then one can prove that $(Hyp(\tau), \circ_h, \sigma_{id})$ is a monoid, where $\sigma_{id}(f_i) = f_i(x_1, x_2, ..., x_{m_i})$ is the identity element, for more detail, see [2].

In 1973, Mal'cev[5] introduced the concept of algebraic systems as follow.

Definition 1. [5] Let I and J be indexed sets. An algebraic system of type (τ, τ') is a triple $(A, (f_i^A)_{i \in I}, (\gamma_j^A)_{j \in J})$ consisting of a nonempty set A, a sequence $(f_i^A)_{i \in I}$ of operations defined on A and a sequence $(\gamma_j^A)_{j \in J}$ of relations on A, where $\tau = (m_i)_{i \in I}$ is a sequence of the arity of each operation f_i^A and $\tau' = (n_j)_{j \in J}$ is a sequence of the arity of each relation γ_j^A. The pair (τ, τ') is called the type of an algebraic system.

In 2008, K. Denecke and D. Phusanga introduced the concept of a hypersubstitution for algebraic systems which is a mapping that assigns an operation symbol to a term and assigns a relation symbol to a formula which preserve the arity. The set of all hypersubstitutions for algebraic systems of type $(\tau, \tau^{'})$ is denoted by $Hyp(\tau, \tau^{'})$. They defined an associative operation $\circ_h$ on this set and proved that $(Hyp(\tau, \tau^{'}), \circ_h, \sigma_{id})$ forms a monoid where σ_{id} is an identity hypersubstitution for algebraic systems, see more detail [3, 6, 8].

The monoid of relational hypersubstitutions for algebraic systems

Any relational hypersubstitution for algebraic systems is a mapping that assigns an operation symbol to a term and assigns a relation symbol to a relational term which preseves the arity.

Definition 2. [6] An $n-$ary quantifier free formular of type $(\tau, \tau^{'})$ is defined as follow. Let j be an indexed set. If $j \in J$ and $t_1, t_2, ..., t_{n_j}$ are $n-$ary terms of type τ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, t_2, ..., t_{n_j})$ is an $n-$ary relational term of type $(\tau, \tau^{'})$.

Let $\gamma F_{(\tau,\tau')}(X_n)$ be the set of all $n-$ary relational term of type $(\tau, \tau^{'})$ and let $\gamma F_{(\tau,\tau')}(X) := \cup_{n\in\mathbb{N}} \gamma F_{(\tau,\tau')}(X_n)$ be the set of all relational terms of type $(\tau, \tau^{'})$.

A relational hypersubstitution for algebraic systems of type $(\tau, \tau^{'})$ is a mapping

$$\sigma : \{f_i |\ i \in I\} \cup \{\gamma_j |\ j \in J\} \to W_\tau(X) \cup \gamma F_(\tau, \tau^{'})(X)$$

with $\sigma(f_i) \in W_\tau(X_{n_i})$ and $\sigma(\gamma_j) \in \gamma F_{(\tau,\tau')}(X_{n_j})$. The set of all relational hypersubstitutions for algebraic systems of type $(\tau, \tau^{'})$ is denoted by $Relhyp(\tau, \tau^{'})$. To defined a binary operation on this set, we give the concept of superposition of relational terms. A superposition of relational terms $R^m_n : (W_\tau(X) \cup \gamma F_{(\tau,\tau')}(X_m)) \times (W_\tau(X_n))^m \to W_\tau(X) \cup \gamma F_{(\tau,\tau')}(X_m)$ is defined by the following steps, for $t, t_1, \ldots, t_{m_i} \in W_\tau(X_m)$, $s_1, \ldots, s_m \in W_\tau(X_n)$,

(i) $R^m_n(t, s_1, \ldots, s_m) := S^m_n(t, s_1, \ldots, s_m)$,

(ii) $R^m_n(F, s_1, \ldots, s_m) := \gamma_j(S^m_n(t_1, s_1, \ldots, s_m), \ldots, S^m_n(t_{n_j}, s_1, \ldots, s_m))$.

Every relational hypersubstitution for algebraic systems σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \cup \gamma\mathcal{F}_{(\tau,\tau')}(X) \to W_\tau(X) \cup \gamma\mathcal{F}_{(\tau,\tau')}(X)$ defined by the following steps.

(i) $\widehat{\sigma}[x_i] := x_i \in X$,

(ii) $\widehat{\sigma}[f_i(t_1 \ldots, t_{m_i})] := S^{m_i}_m(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{m_i}])$,
where $i \in I$ and $t_1, \ldots, t_{m_i} \in W_\tau(X_m)$, i.e., any occurrence of the variable x_k in $\sigma(f_i)$ is replaced by the term $\widehat{\sigma}[t_k], 1 \leq k \leq m_i$,

(iii) $\widehat{\sigma}[\gamma_j(s_1 \ldots, s_{n_j})] := R_n^{n_j}(\sigma(\gamma_j), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_{n_j}])$, where $j \in J$ and $s_1, \ldots, s_{n_j} \in W_\tau(X_n)$, i.e., any occurrence of the variable x_k in $\sigma(\gamma_j)$ is replaced by the term $\widehat{\sigma}[s_k], 1 \leq k \leq n_j$.

They defined a binary operation $\circ_r$ on $Relhyp(\tau, \tau^{'})$ by $\sigma \circ_r \alpha := \hat{\sigma} \circ \alpha$; for all $\alpha, \sigma \in Relhyp(\tau, \tau')$ where "$\circ$" is the usual composition of mappings and $\sigma, \alpha \in Relhyp(\tau, \tau^{'})$. Let σ_{id} be the relational hypersubstitution which maps each m_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{m_i})$ and maps each n_j-ary relation symbol γ_j to the relational term $\gamma_j(x_1, \ldots, x_{n_j})$. D. Phusanga and J. Koppitz [6] proved that $(Relhyp(\tau, \tau^{'}), \circ_r, \sigma_{id})$ is a monoid.

In 2015, W. Wongpinit and S. Leeratanavalee [**?**] introduced the concept of the $i - most$ of terms.

Definition 3. For a type $\tau = (m)$ with an m-ary operation symbol f, $t \in W_{(m)}(X)$ and $1 \leq i \leq m$. An $i - most(t)$ is defined inductively by the following steps.

(i) If t is a variable, then $i - most(t) = t$.

(ii) If $t = f(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n \in W_{(m)}(X)$, then $i - most(t) := i - most(t_i)$.

Example 1. *Let* $\tau = (3)$ *be a type,* $t = f(x_2, f(x_3, x_1, x_2), f(x_2, x_1, x_3))$. *Then* $1 - most(t) = x_2$, $2 - most(t) = 2 - most(f(x_3, x_1, x_2)) = x_1$ *and* $3 - most(t) = 3 - most(f(x_2, x_1, x_3)) = x_3$.

Main Results

Let $(\tau, \tau^{'}) = ((m), (n))$ be a type with an $m-$ary operation symbol f, an $n-$ary relation symbol γ, $t \in W_{(m)}(X_m)$ and $F \in \gamma\mathcal{F}_{((m),(n))}(X_n)$, we denote

$\sigma_{t,F}$:= the relational hypersubstitution for algebraic systems of type $((m), (n))$ with maps f to the term $t \in W_{(m)}(X_m)$ and maps γ to the relational term $F \in \gamma\mathcal{F}_{((m),(n))}(X_n)$,

$var(t)$:= the set of all variables occurring in the term t,

$var(F)$:= the set of all variables occurring in the relational term F,

$leftmost(t)$:= the first variable (from the left) occurring in the term t,

$rightmost(t)$:= the last variable (from the left) occurring in the term t,

$leftmost(F)$:= the first variable (from the left) occurring in the relational term F,

$rightmost(F)$:= the last variable (from the left) occurring in the relational term F.

Let $\sigma_{t,F} \in Relhyp((m), (n))$, we denote

$R'_X := \{\sigma_{t,F} \,|t = x_i \in X_m$ and $F = \gamma(s_1, ..., s_n)$ with $var(F) = \{x_{b_1}, ..., x_{b_l}\} \subseteq X_n$ such that $i - most(s_{b'_k}) = x_{b_k}$ for all $k = 1, ..., l$ and some distinct $b_1^{'}, ..., b_l^{'} \in \{1, ..., n\}$ where $i \in \{1, ..., m\}\}$;

$R_T := \{\sigma_{t,F} \mid t = f(t_1, \ldots, t_m)$ and $F = \gamma(s_1, ..., s_n)$ with $var(t) = \{x_{a_1}, ..., x_{a_k}\}$
and $var(F) = \{x_{b_1}, ..., x_{b_l}\}$ such that $t_{a'_i} = x_{a_i}$ and $s_{b'_j} = x_{b_j}$for all $i = 1, ..., k, j = 1, ..., l$ for some distinct$a'_1, ..., a'_k \in \{1, ..., m\}$ and for some distinct $b'_1, ..., b'_l \in \{1, ..., n\}\}$.

In [4], the authors showed that $R'_X \cup R_T$ is the set of all unit-regular elements in $Relhyp((m), (n))$.

All Maximal Unit-Regular Submonoids of $Relhyp((2), (2))$

Let $(\tau, \tau') = ((2), (2))$ be a type with a biary operation symbol f, a binary relation symbol γ, $t \in W_{(2)}(X_2)$ and $F \in \gamma F_{((2),(2))}(X_2)$. Let $\sigma_{t,F} \in Relhyp((2), (2))$, we denote

$R'_X := \{\sigma_{t,F} \mid t = x_i \in X_2$ and $F = \gamma(s_1, s_2)$ with $var(F) \subseteq X_2$ such that $i-most(s_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}\}$;

$R_T := \{\sigma_{t,F} \mid t = f(t_1, t_2)$ and $F = \gamma(s_1, s_2)$ with $var(t) \subseteq X_2$ and $var(F) \subseteq X_2$ such that $t_{a'_i} = x_{a_i}$ and $s_{b'_j} = x_{b_j}$for all $i, j = 1, 2$ for some distinct $a'_1, a'_2 \in \{1, 2\}$ and for some distinct $b'_1, b'_2 \in \{1, 2\}\}$.

It is easily to see that R'_X, R_T are pairwise disjoint but $\underline{R'_X}, \underline{R_T}$ need not be submonoids of $\underline{Relhyp((2), (2))}$ as the following example.

Example 2. *Let $\sigma_{t,F}, \sigma_{u,H} \in R'_X$ such that $t = x_1, F = \gamma(f(x_2, x_2), f(x_1, x_1))$ and $u = x_2, H = \gamma(f(x_1, x_2), f(x_2, x_1))$. Consider*

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_1,F}[x_2] = x_2,$$

and

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{x_1,F}[h_1], \widehat{\sigma}_{x_1,F}[h_2]) = R_2^2(F, x_1, x_2)\\ &= \gamma(f(x_2, x_2), f(x_1, x_2)).\end{aligned}$$

So $\sigma_{t,F} \circ_r \sigma_{u,H} \notin R'_X$.

Example 3. *Let $\sigma_{t,F}, \sigma_{u,H} \in R_T$ such that $t = f(f(x_1, x_1), x_1), F = \gamma(f(x_2, x_2), x_2)$ and $u = f(f(x_2, x_2), x_2), H = \gamma(x_1, f(x_1, x_1))$. Consider*

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= S_2^2(t, \widehat{\sigma}_{t,F}[u_1], \widehat{\sigma}_{t,F}[x_2])\\ &= S_2^2(t, f(f(x_2, x_2), x_2), x_2)\\ &= f(f(f(f(x_2, x_2), x_2), f(f(x_2, x_2), x_2)), f(f(x_2, x_2), x_2)).\end{aligned}$$

So $\sigma_{t,F} \circ_r \sigma_{u,H} \notin R_T$.

Next, let $\sigma_{t,F} \in Relhyp((2), (2))$, we denote

$R'_{x_i} := \{\sigma_{t,F} \mid t = x_i \in X_2, F = \gamma(s_1, s_2)$ where $var(F) \subseteq X_2$ such that $i-$

$most(s_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}\}$;

$R''_{x_i} := \{\sigma_{t,F} \mid t = x_i \in X_2, F = \gamma(s_1, s_2)$ where $var(F) = \{x_1, x_2\}$ such that $i-most(s_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}$ with if $i = 1$, then $rightmost(s_1) \neq rightmost(s_2)$, if $i = 2$, then $leftmost(s_1) \neq leftmost(s_2)\}$;

$R'''_{x_i} := \{\sigma_{t,F} \mid t = x_i \in X_2, F = \gamma(s_1, s_2)$ where $|var(F)| = 1\}$;

$R_{T_1} := \{\sigma_{t,F} \mid t = f(x_1, x_2), F = \gamma(s_1, s_2)$ where $var(F) \subseteq X_2$ such that $s_{b'_j} = x_{b_j}$ for all $j = 1, 2$ and for some distinct $b'_1, b'_2 \in \{1, 2\}\}$;

$R_{T_2} := \{\sigma_{t,F} \mid t = f(x_2, x_1), F = \gamma(s_1, s_2)$ where $var(F) \subseteq X_2$ such that $s_{b'_j} = x_{b_j}$ for all $j = 1, 2$ and for some distinct $b'_1, b'_2 \in \{1, 2\}\}$;

$R_{T_3} := \{\sigma_{t,F} \mid t = f(x_1, t_2), F = \gamma(x_1, s_2)$ where $|var(t)| = 1$ and $var(F) \subseteq X_2$ such that $s_{b'_j} = x_{b_j}$ for all $j = 1, 2$ and for some distinct $b'_1, b'_2 \in \{1, 2\}\}$;

$R_{T_4} := \{\sigma_{t,F} \mid t = f(t_1, x_2), F = \gamma(s_1, x_2)$ where $|var(t)| = 1$ and $var(F) \subseteq X_2$ such that $s_{b'_j} = x_{b_j}$ for all $j = 1, 2$ and for some distinct $b'_1, b'_2 \in \{1, 2\}\}$;

$R_{T_5} := \{\sigma_{t,F} \mid t = f(t_1, x_1), F = \gamma(s_1, x_1)$ where $|var(t)| = 1$ and $var(F) \subseteq X_2$ such that $s_{b'_j} = x_{b_j}$ for all $j = 1, 2$ and for some distinct $b'_1, b'_2 \in \{1, 2\}\}$;

$R_{T_6} := \{\sigma_{t,F} \mid t = f(x_2, t_2), F = \gamma(x_2, s_2)$ where $|var(t)| = 1$ and $var(F) \subseteq X_2$ such that $s_{b'_j} = x_{b_j}$ for all $j = 1, 2$ and for some distinct $b'_1, b'_2 \in \{1, 2\}\}$.

It is easily to see that R_{T_i} for $i \in \{1, ..., 6\}$ are pairwise disjoint but $\underline{R_{T_i}}$ need not be a submonoid of $\underline{Relhyp((2), (2))}$ as the following example.

Example 4. *Let* $\sigma_{t,F}, \sigma_{u,H} \in R_{T_5}$ *such that* $t = f(f(x_1, x_1), x_1), F = \gamma(x_1, f(x_1, x_1))$ *and* $u = f(f(x_1, x_1), x_1), H = \gamma(f(x_2, x_2), x_2)$. *Then*

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= S_2^2(t, \widehat{\sigma}_{t,F}[u_1], \widehat{\sigma}_{t,F}[x_2]) \\ &= S_2^2(t, f(f(x_1, x_1), x_1), x_1) \\ &= f(f(f(f(x_1, x_1), x_1), f(f(x_1, x_1), x_1)), f(f(x_1, x_1), x_1)), \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2]) \\ &= S_2^2(t, f(f(x_2, x_2), x_2), x_2) \\ &= \gamma(f(f(f(x_2, x_2), x_2), f(f(x_2, x_2), x_2)), f(f(x_2, x_2), x_2)).\end{aligned}$$

So $\sigma_{t,F} \circ_r \sigma_{u,H} \notin R_{T_5}$. *If* F, H *are another case, we can show similar to the previous solution.*

By Example 4., we get that R_{T_5}, R_{T_6} are not closed into itself. Next, let $\sigma_{t,F} \in Relhyp((2), (2))$, we denote

$R^*_{T_i} := \{\sigma_{t,F} \mid t = f(t_1, t_2), F = \gamma(s_1, s_2)$ where $t_i = x_i, s_i = x_i; i = 1, 2$ or $t_i, s_i \in X_2$ such that $|var(t)| = 1$ and $var(t) = var(F)\}$;

$R'_{T_2} := \{\sigma_{t,F} \mid t = f(x_2, x_1), F = \gamma(x_2, x_1)\}$;

$R'_{T_3} := \{\sigma_{t,F} \mid t = f(x_1, t_2), F = \gamma(x_1, s_2)$ where $|var(t)| = 1, |var(F)| = 1$ such that $t_2, s_2 \in W_{(2)}(X_2) \backslash X_2\}$;

$R'_{T_4} := \{\sigma_{t,F} \mid t = f(t_1, x_2), F = \gamma(s_1, x_2)$ where $|var(t)| = 1, |var(F)| = 1$ such that $t_1, s_1 \in W_{(2)}(X_2)\backslash X_2\}$.

We denote $(MUR_1) := R''_{x_i} \cup R'''_{x_i} \cup R^*_{T_i} \cup R'_{T_2}$, $(MUR_2) := R'_{x_1} \cup R'''_{x_i} \cup R^*_{T_1} \cup R'_{T_3}$, $(MUR_3) := R'_{x_2} \cup R'''_{x_i} \cup R^*_{T_1} \cup R'_{T_3}$, $(MUR_4) := R'_{x_1} \cup R'''_{x_i} \cup R^*_{T_2} \cup R'_{T_4}$ and $(MUR_5) := R'_{x_2} \cup R'''_{x_i} \cup R^*_{T_2} \cup R'_{T_4}$.

Proposition 1. *$R''_{x_i} \cup R'''_{x_i} \cup R^*_{T_i}$ is a submonoid of $\underline{Relhyp((2),(2))}$.*

Proof. We show that $R''_{x_i} \cup R'''_{x_i} \cup R^*_{T_i}$ is closed under $\circ_r$.

Case 1: $\sigma_{t,F} \in R''_{x_i}$. Then $t = x_i \in X_2, F = \gamma(s_1, s_2)$ where $var(F) = \{x_1, x_2\}$ such that $i{-}most(s_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}$ with if $i = 1$, then $rightmost(s_1) \neq rightmost(s_2)$, if $i = 2$, then $leftmost(s_1) \neq leftmost(s_2)$.

Case 1.1: $\sigma_{u,H} \in R''_{x_i}$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R^2_2(F, \widehat{\sigma}_{x_i,F}[h_1], \widehat{\sigma}_{x_i,F}[h_2]) \\ &= \gamma(S^2_2(s_1, i-most(h_1), i-most(h_2)), \\ &\quad S^2_2(s_2, i-most(h_1), i-most(h_2))) \\ &= \gamma(s'_1, s'_2) \text{ where } var(\gamma(s'_1, s'_2)) = \{x_{i_1}, x_{i_2}\} \text{ such that} \\ &\quad i-most(s'_{i'_k}) = x_{i_k}; i, k = 1, 2.\end{aligned}$$

Case 1.2: $\sigma_{u,H} \in R'''_{x_i}$. Then $u = x_i \in X_2, H = \gamma(h_1, h_2)$ where $|var(H)| = 1$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R^2_2(F, \widehat{\sigma}_{x_i,F}[h_1], \widehat{\sigma}_{x_i,F}[h_2]) \\ &= \gamma(S^2_2(s_1, i-most(h_1), i-most(h_2)), \\ &\quad S^2_2(s_2, i-most(h_1), i-most(h_2))) \\ &= \gamma(s'_1, s'_2) \text{ where } |var(\gamma(s'_1, s'_2))| = 1.\end{aligned}$$

Case 1.3: $\sigma_{u,H} \in R'_{T_i}$. Then $u = f(u_1, u_2), H = \gamma(h_1, h_2)$ where $u_i, h_i \in X_2$ such that $|var(u)| = 1$ and $var(u) = var(H)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[f(u_1, u_2)] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R^2_2(F, \widehat{\sigma}_{x_i,F}[x_j], \widehat{\sigma}_{x_i,F}[x_j]) \\ &= \gamma(s'_1, s'_2) \text{ where } var(\gamma(s'_1, s'_2)) = \{x_j\}; j = 1, 2.\end{aligned}$$

<u>Case 2</u>: $\sigma_{t,F} \in R'''_{x_i}$. Then $t = x_i \in X_2, F = \gamma(s_1, s_2)$ where $|var(F)| = 1$.

<u>Case 2.1</u>: $\sigma_{u,H} \in R''_{x_i}$. Then $u = x_i \in X_2, H = \gamma(h_1, h_2)$ where $var(H) = \{x_1, x_2\}$ such that $i{-}most(h_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}$ with if $i = 1$, then $rightmost(h_1) \neq rightmost(h_2)$, if $i = 2$, then $leftmost(h_1) \neq leftmost(h_2)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R^2_2(F, \widehat{\sigma}_{x_i,F}[h_1], \widehat{\sigma}_{x_i,F}[h_2]) \\ &= \gamma(S^2_2(s_1, i - most(h_1), i - most(h_2)), \\ &S^2_2(s_2, i - most(h_1), i - most(h_2))) \\ &= \gamma(s'_1, s'_2) \text{ where } |var(\gamma(s'_1, s'_2))| = 1.\end{aligned}$$

<u>Case 2.2</u>: $\sigma_{u,H} \in R'''_{x_i}$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R^2_2(F, \widehat{\sigma}_{x_i,F}[h_1], \widehat{\sigma}_{x_i,F}[h_2]) \\ &= \gamma(S^2_2(s_1, i - most(h_1), i - most(h_2)), \\ &S^2_2(s_2, i - most(h_1), i - most(h_2))) \\ &= \gamma(s'_1, s'_2) \text{ where } |var(\gamma(s'_1, s'_2))| = 1.\end{aligned}$$

<u>Case 2.3</u>: $\sigma_{u,H} \in R^*_{T_i}$. Then $u = f(u_1, u_2), H = \gamma(h_1, h_2)$ where $u_i, h_i \in X_2$ such that $|var(u)| = 1$ and $var(u) = var(H)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[f(u_1, u_2)] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R^2_2(F, \widehat{\sigma}_{x_i,F}[x_j], \widehat{\sigma}_{x_i,F}[x_j]) \\ &= \gamma(s'_1, s'_2) \text{ where } var(\gamma(s'_1, s'_2)) = \{x_j\}; j = 1, 2.\end{aligned}$$

<u>Case 3</u>: $\sigma_{t,F} \in R^*_{T_i}$. Then $t = f(t_1, t_2), F = \gamma(s_1, s_2)$ where $t_i, s_i \in X_2$ such that
$|var(t)| = 1$ and $var(t) = var(F)$.

<u>Case 3.1</u>: $\sigma_{u,H} \in R''_{x_i}$. Then $u = x_i \in X_2, H = \gamma(h_1, h_2)$ where $var(H) = \{x_1, x_2\}$ such that $i{-}most(h_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}$ with if $i = 1$, then $rightmost(h_1) \neq rightmost(h_2)$, if $i = 2$, then $leftmost(h_1) \neq leftmost(h_2)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2]) \\ &= \gamma(S_2^2(s_1, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2]), S_2^2(s_2, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2])) \\ &= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.\end{aligned}$$

Case 3.2: $\sigma_{u,H} \in R'''_{x_i}$. Then $u = x_i \in X_2, H = \gamma(h_1, h_2)$ where $|var(H)| = 1$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2]) \\ &= \gamma(S_2^2(s_1, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2]), S_2^2(s_2, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2])) \\ &= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.\end{aligned}$$

Case 3.3: $\sigma_{u,H} \in R^*_{T_i}$. Then $u = f(u_1, u_2), H = \gamma(h_1, h_2)$ where $u_i, h_i \in X_2$ such that $|var(u)| = 1$ and $var(u) = var(H)$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= \widehat{\sigma}_{t,F}[f(u_1, u_2)] \\ &= f(t_1', t_2') \text{ where } var(\gamma(t_1', t_2')) = \{x_j\}; j = 1, 2, \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{t,F}[x_j], \widehat{\sigma}_{t,F}[x_j]) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_j\}; j = 1, 2.\end{aligned}$$

Then $\sigma_{t,F} \circ_r \sigma_{u,H}, \sigma_{u,H} \circ_r \sigma_{t,F} \in R''_{x_i} \cup R'''_{x_i} \cup R^*_{T_i}$ and $R''_{x_i} \cup R'''_{x_i} \cup R^*_{T_i}$ is a submonoid of $\underline{Relhyp((2),(2))}$. □

Proposition 2. *$R'_{x_1} \cup R'''_{x_i} \cup R^*_{T_i}$ and $R'_{x_2} \cup R'''_{x_i} \cup R^*_{T_i}$ are submonoids of $\underline{Relhyp((2),(2))}$.*

Proof. We show that $R'_{x_1} \cup R'''_{x_i} \cup R^*_{T_i}$ is closed under $\circ_r$.

Case 1: $\sigma_{t,F} \in R'_{x_1}$. Then $t = x_1, F = \gamma(s_1, s_2)$ where $var(F) \subseteq X_2$ such that $i-most(s_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}$.

Case 1.1: $\sigma_{u,H} \in R'_{x_1}$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_1] = x_1, \text{ and}$$

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = \widehat{\sigma}_{t,F}[\gamma(h_1, h_2)].$$

(1) If $|var(F)| = 1$, $|var(H)| = 1$, then

$$\begin{aligned}\widehat{\sigma}_{t,F}[\gamma(h_1, h_2)] &= R_2^2(F, \widehat{\sigma}_{x_1,F}[h_1], \widehat{\sigma}_{x_1,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.\end{aligned}$$

(2) If $|var(F)| = 1, |var(H)| = 2$, then

$$\begin{aligned}\widehat{\sigma}_{t,F}[\gamma(h_1,h_2)] &= R_2^2(F, \widehat{\sigma}_{x_1,F}[h_1], \widehat{\sigma}_{x_1,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.\end{aligned}$$

(3) If $|var(F)| = 2, |var(H)| = 1$, then

$$\begin{aligned}\widehat{\sigma}_{t,F}[\gamma(h_1,h_2)] &= R_2^2(F, \widehat{\sigma}_{x_1,F}[h_1], \widehat{\sigma}_{x_1,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.\end{aligned}$$

(4) If $|var(F)| = 2, |var(H)| = 2$, then

$$\begin{aligned}\widehat{\sigma}_{t,F}[\gamma(h_1,h_2)] &= R_2^2(F, \widehat{\sigma}_{x_1,F}[h_1], \widehat{\sigma}_{x_1,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_{b_1}, x_{b_2}\} \\ &\quad \text{such that } 1-most(s_{b_k'}) = x_{b_k} \text{ for all } i,k = 1,2.\end{aligned}$$

Case 1.2: $\sigma_{u,H} \in R'''_{x_i}$. Then $u = x_i \in X_2, H = \gamma(h_1,h_2)$ where $|var(H)| = 1$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{x_i,F}[h_1], \widehat{\sigma}_{x_i,F}[h_2]) \\ &= \gamma(S_2^2(s_1, i-most(h_1), i-most(h_2)), \\ &\quad S_2^2(s_2, i-most(h_1), i-most(h_2))) \\ &= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.\end{aligned}$$

Case 1.3: $\sigma_{u,H} \in R^*_{T_i}$. Then $u = f(u_1,u_2), H = \gamma(h_1,h_2)$ where $u_i, h_i \in X_2$ such that $|var(u)| = 1$ and $var(u) = var(H)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[f(u_1,u_2)] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{x_i,F}[x_j], \widehat{\sigma}_{x_i,F}[x_j]) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_j\}; j = 1,2.\end{aligned}$$

Case 2: $\sigma_{t,F} \in R'''_{x_i}$. Then $t = x_i \in X_2, F = \gamma(s_1,s_2)$ where $|var(F)| = 1$.
Case 2.1: $\sigma_{u,H} \in R'_{x_1}$. Then $u = x_1, H = \gamma(h_1,h_2)$ where $var(H) \subseteq X_2$ such that
$i-most(h_{b_k'}) = x_{b_k}$ for all $i,k = 1,2$ and some distinct $b_1', b_2' \in \{1,2\}$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_1] = x_1, \text{ and}$$

$$
\begin{aligned}
(\sigma_{t,F}\circ_r \sigma_{u,H})(\gamma) &= R_2^2(F,\widehat{\sigma}_{x_i,F}[h_1],\widehat{\sigma}_{x_i,F}[h_2])\\
&= \gamma(S_2^2(s_1, i-most(h_1), i-most(h_2)),\\
&S_2^2(s_2, i-most(h_1), i-most(h_2)))\\
&= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.
\end{aligned}
$$

Case 2.2: $\sigma_{u,H}\in R_{x_i}'''$. The proof is similar to case 2.2 of Proposition 1.

Case 2.3: $\sigma_{u,H}\in R_{T_i}^{*}$. The proof is similar to case 2.3 of Proposition 1.

Case 3: $\sigma_{t,F}\in R_{T_i}^{*}$. Then $t=f(t_1,t_2), F=\gamma(s_1,s_2)$ where $t_i, s_i\in X_2$ such that $|var(t)|=1$ and $var(t)=var(F)$.

Case 3.1: $\sigma_{u,H}\in R_{x_1}'$. Then $u=x_1, H=\gamma(h_1,h_2)$ where $var(H)\subseteq X_2$ such that $i-most(h_{b_k'})=x_{b_k}$ for all $i,k=1,2$ and some distinct $b_1', b_2'\in\{1,2\}$. Consider

$$(\sigma_{t,F}\circ_r \sigma_{u,H})(f)=\widehat{\sigma}_{t,F}[x_1]=x_1, \text{ and}$$

$$
\begin{aligned}
(\sigma_{t,F}\circ_r \sigma_{u,H})(\gamma) &= R_2^2(F,\widehat{\sigma}_{t,F}[h_1],\widehat{\sigma}_{t,F}[h_2])\\
&= \gamma(S_2^2(s_1,\widehat{\sigma}_{t,F}[h_1],\widehat{\sigma}_{t,F}[h_2]), S_2^2(s_2,\widehat{\sigma}_{t,F}[h_1],\widehat{\sigma}_{t,F}[h_2]))\\
&= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.
\end{aligned}
$$

Case 3.2: $\sigma_{u,H}\in R_{x_i}'''$. The proof is similar to case 3.2 of Proposition 1.

Case 3.3: $\sigma_{u,H}\in R_{T_i}^{*}$. The proof is similar to case 3.3 of Proposition 1.

Therefore $\sigma_{t,F}\circ_r\sigma_{u,H}, \sigma_{u,H}\circ_r\sigma_{t,F}\in R_{x_1}'\cup R_{x_i}'''\cup R_{T_i}^{*}$. For $R_{x_2}'\cup R_{x_i}'''\cup R_{T_i}^{*}$, the proof is similar to the previous proof. □

Theorem 1. *$\underline{(MUR_1)}$ is a unit-regular submonoid of $\underline{Relhyp((2),(2))}$.*

Proof. We get that every element in (MUR_1) is unit-regular. Next we show that $(MUR_1)=R_{x_i}''\cup R_{x_i}'''\cup R_{T_i}^{*}\cup R_{T_2}'$ is closed under $\circ_r$. By Proposition 1, we have $R_{x_i}''\cup R_{x_i}'''\cup R_{T_i}^{*}$ is a submonoid of *Relhyp*$((2),(2))$. So we consider some cases in (MUR_1). Let $\sigma_{t,F},\sigma_{u,H}\in(MUR_1)$.

Case 1: $\sigma_{t,F}\in(MUR_1)$ and $\sigma_{u,H}\in R_{T_2}'$. Then $u=f(x_2,x_1), H=\gamma(x_2,x_1)$.

Case 1.1: $\sigma_{t,F}\in R_{x_i}''$. Then $t=x_i\in X_2, F=\gamma(s_1,s_2)$ where $var(F)=\{x_1,x_2\}$ such that $i-most(s_{b_k'})=x_{b_k}$ for all $i,k=1,2$ and some distinct $b_1', b_2'\in\{1,2\}$ with if $i=1$, then $rightmost(s_1)\neq rightmost(s_2)$, if $i=2$, then $leftmost(s_1)\neq leftmost(s_2)$. Consider

$$(\sigma_{t,F}\circ_r \sigma_{u,H})(f)=\widehat{\sigma}_{x_i,F}[f(x_2,x_1)]=\begin{cases} x_2 & \text{if } i=1\\ x_1 & \text{if } i=2\end{cases}, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{x_i,F}[x_2], \widehat{\sigma}_{x_i,F}[x_1]) \\ &= \gamma(S_2^2(s_1, x_2, x_1), S_2^2(s_2, x_2, x_1)) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_{i_1}, x_{i_2}\} \text{ such that} \\ & i-most(s_{i_k'}') = x_{i_k}; i, k = 1, 2.\end{aligned}$$

Case 1.2: $\sigma_{t,F} \in R_{x_i}'''$. Then $t = x_i \in X_2, F = \gamma(s_1, s_2)$ where $|var(F)| = 1$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[f(x_2, x_1)] = \begin{cases} x_2 & \text{if } i = 1 \\ x_1 & \text{if } i = 2 \end{cases}, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{x_i,F}[h_1], \widehat{\sigma}_{x_i,F}[h_2]) \\ &= \gamma(S_2^2(s_1, x_2, x_1), S_2^2(s_2, x_2, x_1)) \\ &= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.\end{aligned}$$

Case 1.3: $\sigma_{t,F} \in R_{T_i}^*$. Then $t = f(t_1, t_2), F = \gamma(s_1, s_2)$ where $t_i, s_i \in X_2$ such that
$|var(t)| = 1$ and $var(t) = var(F)$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= S_2^2(t, \widehat{\sigma}_{t,F}[x_2], \widehat{\sigma}_{t,F}[x_1]) \\ &= f(t_1', t_2') \text{ where } var(f(t_1', t_2')) = \{x_j\}; j = 1, 2, \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{t,F}[x_2], \widehat{\sigma}_{t,F}[x_1]) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_j\}; j = 1, 2.\end{aligned}$$

Case 2: $\sigma_{t,F} \in R_{T_2}'$. Then $t = f(x_2, x_1), F = \gamma(x_2, x_1)$.

Case 2.1: $\sigma_{u,H} \in R_{x_i}''$. Then $u = x_i \in X_2, H = \gamma(h_1, h_2)$ where $var(H) = \{x_1, x_2\}$ such that $i{-}most(h_{b_k'}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b_1', b_2' \in \{1, 2\}$ with if $i = 1$, then $rightmost(h_1) \neq rightmost(h_2)$, if $i = 2$, then $leftmost(h_1) \neq leftmost(h_2)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_{i_1}, x_{i_2}\} \text{ such that} \\ & i-most(s_{i_k'}') = x_{i_k}; i, k = 1, 2.\end{aligned}$$

Case 2.2: $\sigma_{u,H} \in R_{x_i}'''$. Then $u = x_i \in X_2, H = \gamma(h_1, h_2)$ where $|var(H)| = 1$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.\end{aligned}$$

Case 2.3: $\sigma_{u,H} \in R^*_{T_i}$. Then $u = f(u_1, u_2), H = \gamma(h_1, h_2)$ where $u_i, h_i \in X_2$ such that $|var(u)| = 1$ and $var(u) = var(H)$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= S_2^2(t, \widehat{\sigma}_{t,F}[u_1], \widehat{\sigma}_{t,F}[u_2]) \\ &= \gamma(t_1', t_2') \text{ where } var(\gamma(t_1', t_2')) = \{x_j\}; j = 1, 2, \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_j\}; j = 1, 2.\end{aligned}$$

Case 2.4: $\sigma_{u,H} \in R'_{T_2}$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= S_2^2(t, \widehat{\sigma}_{t,F}[x_2], \widehat{\sigma}_{t,F}[x_1]) \\ &= f(x_1, x_2), \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{t,F}[x_2], \widehat{\sigma}_{t,F}[x_1]) \\ &= \gamma(x_1, x_2).\end{aligned}$$

Therefore $\underline{(MUR_1)}$ is a unit-regular submonoid of $\underline{Relhyp((2),(2))}$. □

Theorem 2. *$\underline{(MUR_2)}$, $\underline{(MUR_3)}$ are unit-regular submonoids of $\underline{Relhyp((2),(2))}$.*

Proof. We get that every element in (MUR_2) is unit-regular. Next we show that $(MUR_2) = R'_{x_1} \cup R'''_{x_i} \cup R^*_{T_1} \cup R'_{T_3}$ is closed under $\circ_r$. By Proposition 2, we have $R'_{x_1} \cup R'''_{x_i} \cup R^*_{T_i}$ is a submonoid of *Relhyp*$((2),(2))$. So we consider some cases in (MUR_2). Let $\sigma_{t,F}, \sigma_{u,H} \in (MUR_2)$.

Case 1: $\sigma_{t,F} \in (MUR_2)$ and $\sigma_{u,H} \in R'_{T_3}$. Then $u = f(x_1, u_2), H = \gamma(x_1, h_2)$
where $|var(u)| = 1, |var(H)| = 1$ such that $u_2, h_2 \in W_{(2)}(X_2) \backslash X_2$.

Case 1.1: $\sigma_{t,F} \in R'_{x_1}$. Then $t = x_1, F = \gamma(s_1, s_2)$ where $var(F) \subseteq X_2$ such that
$1 - most(s_{b_k'}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b_1', b_2' \in \{1, 2\}$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_1,F}[f(x_1, u_2)] = x_1, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{x_1,F}[x_1], \widehat{\sigma}_{x_1,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_1\}.\end{aligned}$$

Case 1.2: $\sigma_{t,F} \in R'''_{x_i}$. Then $t = x_i \in X_2, F = \gamma(s_1, s_2)$ where $|var(F)| = 1$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[f(x_1, u_2)] = x_1, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{x_i,F}[x_1], \widehat{\sigma}_{x_i,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_1\}.\end{aligned}$$

Case 1.3: $\sigma_{t,F} \in R^*_{T_1}$. Then $t = f(t_1, t_2), F = \gamma(s_1, s_2)$ where $t_i, s_i \in X_2$ such that
$|var(t)| = 1$ and $var(t) = var(F)$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= S_2^2(t, \widehat{\sigma}_{t,F}[x_1], \widehat{\sigma}_{t,F}[u_2]) \\ &= f(t_1', t_2') \text{ where } var(f(t_1', t_2')) = \{x_1\}, \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_2^2(F, \widehat{\sigma}_{t,F}[x_1], \widehat{\sigma}_{t,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_1\}.\end{aligned}$$

Case 2: $\sigma_{t,F} \in R'_{T_3}$. Then $t = f(x_1, t_2), F = \gamma(x_1, s_2)$ where $|var(t)| = 1, |var(F)| = 1$ such that $t_2, s_2 \in W_{(2)}(X_2)\backslash X_2$.

Case 2.1: $\sigma_{u,H} \in R'_{x_1}$. Then $u = x_1, H = \gamma(h_1, h_2)$ where $var(H) \subseteq X_2$ such that
$1 - most(h_{b_k'}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b_1', b_2' \in \{1, 2\}$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_1] = x_1, \text{ and}$$

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = \widehat{\sigma}_{t,F}[\gamma(h_1, h_2)].$$

(1) If $|var(H)| = 1$, then

$$\begin{aligned}\widehat{\sigma}_{t,F}[\gamma(h_1, h_2)] &= R_2^2(F, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } |var(\gamma(s_1', s_2'))| = 1.\end{aligned}$$

(2) If $|var(H)| = 2$, then

$$\begin{aligned}\widehat{\sigma}_{t,F}[\gamma(h_1, h_2)] &= R_2^2(F, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2]) \\ &= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_{i_1}, x_{i_2}\} \text{ such that} \\ & i - most(s'_{i_k'}) = x_{i_k}; i, k = 1, 2.\end{aligned}$$

Case 2.2: $\sigma_{u,H} \in R'''_{x_i}$. Then $u = x_i \in X_2, H = \gamma(h_1, h_2)$ where $|var(H)| = 1$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = R_2^2(F, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2])$$
$$= \gamma(s'_1, s'_2) \text{ where } var(\gamma(s'_1, s'_2)) = \{x_j\}; j = 1, 2.$$

Case 2.3: $\sigma_{u,H} \in R^*_{T_1}$. Then $u = f(u_1, u_2), H = \gamma(h_1, h_2)$ where $u_i, h_i \in X_2$ such that $|var(u)| = 1$ and $var(u) = var(H)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = S_2^2(t, \widehat{\sigma}_{t,F}[u_1], \widehat{\sigma}_{t,F}[u_2])$$
$$= f(t'_1, t'_2) \text{ where } var(f(t'_1, t'_2)) = \{x_j\}; j = 1, 2, \text{ and}$$

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = R_2^2(F, \widehat{\sigma}_{t,F}[h_1], \widehat{\sigma}_{t,F}[h_2])$$
$$= \gamma(s'_1, s'_2) \text{ where } var(\gamma(s'_1, s'_2)) = \{x_j\}; j = 1, 2.$$

Case 2.4: $\sigma_{u,H} \in R'_{T_3}$. Then $u = f(x_1, u_2), H = \gamma(x_1, h_2)$ where $|var(u)| = 1, |var(H)| = 1$ such that $u_2, h_2 \in W_{(2)}(X_2)\backslash X_2$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = S_2^2(t, \widehat{\sigma}_{t,F}[x_1], \widehat{\sigma}_{t,F}[u_2])$$
$$= f(x_1, t'_2) \text{ where } var(f(x_1, t'_2)) = \{x_1\}, \text{ and}$$

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = R_2^2(F, \widehat{\sigma}_{t,F}[x_1], \widehat{\sigma}_{t,F}[h_2])$$
$$= \gamma(x_1, s'_2) \text{ where } var(\gamma(x_1, s'_2)) = \{x_1\}.$$

Therefore $\underline{(MUR_2)}$ is a unit-regular submonoid of $\underline{Relhyp((2),(2))}$. For $\underline{(MUR_3)}$ is a unit-regular submonoid of $\underline{Relhyp((2),(2))}$, the proof is similar to the previous proof. □

Theorem 3. $\underline{(MUR_4)}$, $\underline{(MUR_5)}$ *are unit-regular submonoids of* $\underline{Relhyp((2),(2))}$.

Proof. $\underline{(MUR_4)}$, $\underline{(MUR_5)}$ are unit-regular submonoids of $\underline{Relhyp((2),(2))}$, the proof is similar to the Theorem 2. proof. □

Theorem 4. $\underline{(MUR_1)}$ *is a maximal unit-regular submonoid of* $\underline{Relhyp((2),(2))}$.

Proof. Let K be a proper unit-regular submonoid of $Relhyp((2),(2))$ such that $(MUR_1) \subseteq K \subset Relhyp((2),(2))$. Let $\sigma_{t,F} \in K$, then $\sigma_{t,F}$ is unit-regular.

Case 1: $\sigma_{t,F} \in R'_{x_i} \backslash R''_{x_i} \cup R'''_{x_i}$. Then $t = x_i \in X_2, F = \gamma(s_1, s_2)$ where $var(F) = \{x_1, x_2\}$ such that $i{-}most(s_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}$ with if $i = 1$, then $rightmost(s_1) = rightmost(s_2)$, if $i = 2$, then $leftmost(s_1) = leftmost(s_2)$.

Case 1.1: $i = 1$. Choose $\sigma_{u,H} \in R''_{x_2}$. Then $u = x_2, H = \gamma(h_1, h_2)$ where $var(H) = \{x_1, x_2\}$ such that $2{-}most(h_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}$ with $leftmost(h_1) \neq leftmost(h_2)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_2] = x_2, \text{ and}$$

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = R_2^2(F, 1-most(h_1), 1-most(h_2))$$
$$= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_{i_1}, x_{i_2}\} \text{ such that}$$
$$rightmost(s_1') = rightmost(s_2').$$

So $\sigma_{u,H} \circ_r \sigma_{t,F} \notin R'_X$.

Case 1.2: $i = 2$. Choose $\sigma_{u,H} \in R''_{x_1}$. Then $u = x_1, H = \gamma(h_1, h_2)$ where $var(H)$ $= \{x_1, x_2\}$ such that $1-most(h_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct b'_1, b'_2 $\in \{1, 2\}$ with $rightmost(h_1) \neq rightmost(h_2)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_1] = x_1, \text{ and}$$

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = R_2^2(F, 2-most(h_1), 2-most(h_2))$$
$$= \gamma(s_1', s_2') \text{ where } var(\gamma(s_1', s_2')) = \{x_{i_1}, x_{i_2}\} \text{ such that}$$
$$leftmost(s_1') = leftmost(s_2').$$

So $\sigma_{u,H} \circ_r \sigma_{t,F} \notin R'_X$.

Case 2: $\sigma_{t,F} \in R_T \backslash R'_{T_1} \cup R'_{T_2}$.

Case 2.1: $\sigma_{t,F} \in R'_{T_3}$. Then $t = f(x_1, t_2), F = \gamma(x_1, s_2)$ where $|var(t)| = 1, |var(F)| = 1$ such that $t_2, s_2 \in W_{(2)}(X_2) \backslash X_2$. Choose $\sigma_{u,H} \in R'_{T_2}$, then $u = f(x_2, x_1)$, $H = \gamma(x_2, x_1)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = S_2^2(t, \widehat{\sigma}_{t,F}[x_2], \widehat{\sigma}_{t,F}[x_1])$$
$$= f(x_2, t_2') \text{ where } var(f(x_2, t_2')) = \{x_2\}, \text{ and}$$

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = R_2^2(F, \widehat{\sigma}_{t,F}[x_2], \widehat{\sigma}_{t,F}[x_1])$$
$$= \gamma(x_2, s_2') \text{ where } var(\gamma(x_2, s_2')) = \{x_2\}.$$

So $\sigma_{t,F} \circ_r \sigma_{u,H} \in R_6$ and is not closed into itself.

Case 2.2: $\sigma_{t,F} \in R'_{T_4}$. Then $t = f(t_1, x_2), F = \gamma(s_1, x_2)$ where $|var(t)| = 1, |var(F)| = 1$ such that $t_1, s_1 \in W_{(2)}(X_2) \backslash X_2$. Choose $\sigma_{u,H} \in R'_{T_2}$. Then $u = f(x_2, x_1)$, $H = \gamma(x_2, x_1)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = S_2^2(t, \widehat{\sigma}_{t,F}[x_2], \widehat{\sigma}_{t,F}[x_1])$$
$$= f(t_1', x_1) \text{ where } var(f(t_1', x_1)) = \{x_1\}, \text{ and}$$

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = R_2^2(F, \widehat{\sigma}_{t,F}[x_2], \widehat{\sigma}_{t,F}[x_1])$$
$$= \gamma(s_1', x_1) \text{ where } var(\gamma(s_1', x_1)) = \{x_1\}.$$

So $\sigma_{t,F} \circ_r \sigma_{u,H} \in R_5$ and is not closed into itself.

Thus $\sigma_{t,F} \in (MUR_1)$. Therefore $K \subseteq (MUR_1)$ and thus $\underline{K} = \underline{(MUR_1)}$. □

Theorem 5. *$\underline{(MUR_2)}, \underline{(MUR_3)}$ are maximal unit-regular submonoids of $\underline{Relhyp((2),(2))}$.*

Proof. Let K be a proper unit-regular submonoid of $Relhyp((2),(2))$ such that $(MUR_2) \subseteq K \subset Relhyp((2),(2))$. Let $\sigma_{t,F} \in K$. Then $\sigma_{t,F}$ is unit-regular.

Case 1: $\sigma_{t,F} \in R'_{x_i} \backslash R'_{x_1} \cup R'''_{x_i}$. Then $t = x_2, F = \gamma(s_1, s_2)$ where $var(F) = \{x_1, x_2\}$ such that $2-most(s_{b'_k}) = x_{b_k}$ for all $i, k = 1, 2$ and some distinct b'_1, b'_2.

Case 1.1: $If\ lefmost(s_1) = leftmost(s_2)$. Choose $\sigma_{u,H} \in R'_{x_1}$. Then $u = x_1, H = \gamma(h_1, h_2)$ where $var(H) = \{x_1, x_2\}$ such that $1 - most(h_{b'_k}) = x_{b_k}$ for all i, k
$= 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}$ with $rightmost(h_1) \neq rightmost(h_2)$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_1] = x_1, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R^2_2(F, 2 - most(h_1), 2 - most(h_2)) \\ &= \gamma(s'_1, s'_2) \text{ where } var(\gamma(s'_1, s'_2)) = \{x_{i_1}, x_{i_2}\} \text{ such that} \\ &\quad leftmost(s'_1) = leftmost(s'_2).\end{aligned}$$

So $\sigma_{u,H} \circ_r \sigma_{t,F} \notin R'_X$.

Case 1.2: $If\ lefmost(s_1) \neq leftmost(s_2)$. Choose $\sigma_{u,H} \in R'_{x_1}$. Then $u = x_1, H = \gamma(h_1, h_2)$ where $var(H) = \{x_1, x_2\}$ such that $1 - most(h_{b'_k}) = x_{b_k}$ for all i, k
$= 1, 2$ and some distinct $b'_1, b'_2 \in \{1, 2\}$ with $rightmost(h_1) = rightmost(h_2)$. Consider

$$(\sigma_{u,H} \circ_r \sigma_{t,F})(f) = \widehat{\sigma}_{u,H}[x_2] = x_2, \text{ and}$$

$$\begin{aligned}(\sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) &= R^2_2(H, 1 - most(s_1), 1 - most(s_2)) \\ &= \gamma(h'_1, h'_2) \text{ where } var(\gamma(h'_1, h'_2)) = \{x_{i_1}, x_{i_2}\} \text{ such that} \\ &\quad rightmost(h'_1) = rightmost(h'_2).\end{aligned}$$

So $\sigma_{u,H} \circ_r \sigma_{t,F} \notin R'_X$.

Case 2: $\sigma_{t,F} \in R_T \backslash R'_{T_1} \cup R'_{T_3}$. Choose $\sigma_{u,H} \in R'_{T_3}$. Then $u = f(x_1, u_2), H = \gamma(x_1, h_2)$ where $|var(u)| = 1, |var(H)| = 1$ such that $u_2, h_2 \in W_{(2)}(X_2) \backslash X_2$.

Case 2.1: $\sigma_{t,F} \in R'_{T_2}$. Then $t = f(x_2, x_1)$, $F = \gamma(x_2, x_1)$. Consider

$$\begin{aligned}(\sigma_{u,H} \circ_r \sigma_{t,F})(f) &= S^2_2(u, \widehat{\sigma}_{u,H}[x_2], \widehat{\sigma}_{u,H}[x_1]) \\ &= f(x_2, u'_2) \text{ where } var(f(x_2, u'_2)) = \{x_2\}, \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) &= R^2_2(H, \widehat{\sigma}_{u,H}[x_2], \widehat{\sigma}_{u,H}[x_1]) \\ &= \gamma(x_2, h'_2) \text{ where } var(\gamma(x_2, h'_2)) = \{x_2\}.\end{aligned}$$

So $\sigma_{u,H} \circ_r \sigma_{t,F} \in R_6$ and is not closed into itself.

Case 2.2: $\sigma_{t,F} \in R'_{T_4}$. Then $t = f(t_1, x_2), F = \gamma(s_1, x_2)$ where $|var(t)| = 1, |var(F)| = 1$ such that $t_1, s_1 \in W_{(2)}(X_2)\backslash X_2$. Consider

$$\begin{aligned}(\sigma_{u,H} \circ_r \sigma_{t,F})(f) &= S_2^2(u, \widehat{\sigma}_{u,H}[t_1], \widehat{\sigma}_{u,H}[x_2]) \\ &= f(u'_1, u'_2) \text{ where } u'_i \in W_{(2)}(X_2)\backslash X_2, \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) &= R_2^2(H, \widehat{\sigma}_{u,H}[s_1], \widehat{\sigma}_{u,H}[x_2]) \\ &= \gamma(h'_1, h'_2) \text{ where } h'_i \in W_{(2)}(X_2)\backslash X_2.\end{aligned}$$

So $\sigma_{u,H} \circ_r \sigma_{t,F}$ is not unit-regular.

Thus $\sigma_{t,F} \in (MUR_2)$. Therefore $K \subseteq (MUR_2)$ and thus $\underline{K} = \underline{(MUR_2)}$. For $\underline{(MUR_3)}$ is a maximal unit-regular submonoid of $\underline{Relhyp((2),(2))}$, the proof is similar to the previous proof. □

Theorem 6. $\underline{(MUR_4)}, \underline{(MUR_5)}$ *are maximal unit-regular submonoids of* $\underline{Relhyp((2),(2))}$.

Proof. $\underline{(MUR_4)}, \underline{(MUR_5)}$ are maximal unit-regular submonoids of $\underline{Relhyp((2),(2))}$, the proof is similar to the Theorem 5 proof. □ In 1980, H.D. Alarcao showed that: A monoid S is factorisable if and only if it is unit-regular[1].

Corollary 1. $(MUR_1), (MUR_2), (MUR_3), (MUR_4), (MUR_5)$ *are maximal factorisable submonoids of the monoid relational hypersubstitutions for algebraic systems of type* $((2),(2))$.

Acknowledgements

The authors are highly grateful to the referees for their valuable comments and suggestions for improving the article. This research was supported by Chiang Mai University, Chiang Mai 50200, Thailand.

References

[1] H. D. Alarcao, Factorisable as a finiteness condition, Agronomy Journal, **20** (1980), 281–282.

[2] K. Denecke and D. Lau and R. Pöschel and D. Schweigert, Hyperidentities, hyperequational classes and clone congruences, Contributions to General Algebra, **7** (Verlag Hölder-Pichler-Tempsky, Austria, 1991), 97–118.

[3] K. Denecke and D. Phusang, Hyperformulas and solid algebraic systems, Studdia logica, **90** (2008), 263–286.

[4] P. Kunama and S. Leeratanavalee, All unit-regular elements of relational hypersubstitutions for algebraic systems, (preprint).

[5] A. I. Mal'cev, Algebraic Systems, Akademie-Verlag, Berlin, Germany, 1973.

[6] D. Phusanga, Derived Algebraic Systems, Ph.D thesis, Universität Potsdam in Brandenburg, 2013.

[7] D. Phusanga and J. Koppitz, The monoid of hypersubstitutions for algebraic systems, Announcements of Union of Scientists Silven, **33** (2018), 119–126.

[8] D. Phusanga, Some varieties of algebraic systems of type $((m),(n))$, Asian European Journal of Mathematics, **12** (2017), 1950005.

[9] Sh. L. Wismath, The monoid of hypersubstitutions of type *(n)*, Southest Asian Bulletin of Mathematics, **12**(1) (2000), 115–128.

[10] W. Wongpinit and S. Leeratanavalee, All maximal idempotent submonoids of $Hyp_G(n)$, Surveys in Mathematics and its Applications, **10** (2015), 41–48.